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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XVIII.

FROM NOVEMBER, 1886, TO NOVEMBER, 1887.

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PROCEEDINGS

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LONDON MATHEMATICAL SOCIETY.

VOL. XVIII.

TWENTY-THIRD SESSION, 1886—1887.

November 11*th*, 1886.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. F. S. McAulay, M.A., Mathematical Master, St. Paul's School, was elected a Member.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by Captain Macmahon, R.A., seconded by Mr. A. B. Basset, and carried.

At the request of the Chairman, Mr. G. Heppel consented to act as Auditor.

From the report of the Secretaries, it appeared that the number of Members since the General Meeting held November 12th, 1885, had increased from 181 to 184, of these 68 being Life Members.

The Society had lost one member by death, viz., Mr. Duncan Brockelbank, who died September 27th, 1885, on board the "Melbourne," during his passage to Australia.

The following communications had been made :—

On Waves Propagated along the Plane Surface of an Elastic Solid: Lord Rayleigh.

On the Application of Clifford's Graphs to Ordinary Binary Quantics: Mr. Kempe.

On Clifford's Theory of Graphs: Mr. Buchheim.

On Unicursal Curves: Mr. R. A. Roberts.

On some Consequences of the Transformation Formula $y = sn(L + A + B + C + \ldots)$: Mr. J. Griffiths.

On the Numerical Solution of Cubic Equations: Mr. Heppel.

On a Theorem in Kinematics: Mr. J. J. Walker.

Note on the Induction of Electric Currents in an Infinite Plane Current Sheet, which is rotating in a field of Magnetic Force: Mr. Basset.

Logarithms in General Logic: Mrs. Bryant.

On a Class of Integrable Reciprocants: Mr. Hammond.

On Perpetuant Reciprocants: Captain Macmahon.

Note on the Functions $\mathrm{Z}(u)$, $\Theta(u)$, $\Pi(u, a)$: Mr. Glaisher.

Note on a $\mathrm{Z}(u)$ Function: Mr. J. Griffiths.

On Polygons Circumscribed about a Conic, and Inscribed in a Cubic: Mr. R. A. Roberts.

On an Instantaneous Proof of the Expression for the Number of Linearly Independent Invariants or Seminvariants of a given Type, and also of the corresponding Expression for Reciprocants: Prof. Sylvester.

On Ternary and n-ary Reciprocants: Mr. Elliott.

Homographic, Circular, and Projective Reciprocants: Mr. L. J. Rogers.

A Proof of Cayley's Fundamental Theorem of Invariants: Captain Macmahon.

Note on the Invariantisers of a Binary Quantic: Mr. J. Griffiths.

On the Number of Linearly Independent Invariants (or Seminvariants), Reciprocants, or in general of Integrals of any Assigned Type, of a Homogeneous and Isobaric Linear Partial Differential Equation: Prof. Sylvester.

On some Results connected with the Theory of Reciprocants: Mr. Leudesdorf.

On Cremonian Congruences contained in Linear Complexes: Dr. Hirst.

Solution of the Cubic and Biquadratic Equation by means of Weierstrass's Elliptic Functions: Prof. Greenhill.

On the Complex of Lines which meet a Unicursal Quartic Curve: Prof. Cayley.

On the Airy-Maxwell Solution of the Equations of Equilibrium of an Isotropic Elastic Solid under Conservative Forces: Mr. Ibbetson.

On the Converse of Stereographic Projection and on Contangential and Coaxal Spherical Circles: Mr. Jeffery.

Reciprocation in Statics: Prof. Genese.

Formula for the Interchange of the Independent and Dependent Variables, with some applications to Reciprocants: Mr. Leudesdorf.

Second Paper on Reciprocants: Mr. Rogers.

On the Theory of Screws in Elliptic Space (Third Note): Mr. Buchheim.

On the Motion of a Liquid Ellipsoid under the influence of its own attraction: Mr. Basset.

Some Applications of Weierstrass's Elliptic Functions: Prof. Greenhill.

Electrical Oscillations on Cylindrical Conductors: Prof. J. J. Thomson.

Minor communications were made by the President, Messrs. Kempe and H. M. Taylor, Captain Macmahon, and the Rev. T. C. Simmons.

The same Journals had been subscribed for as in the preceding Session.

No addition had been made to the List of Exchanges.

The meeting next proceeded to the election of the new Council. The Scrutators (Prof. W. Woolsey Johnson and the Rev. T. R. Terry)

having examined the Balloting Lists, declared the following gentlemen duly elected:—

President, Sir James Cockle, F.R.S.; Vice-Presidents, J. W. L. Glaisher, F.R.S., Prof. Hart, Lord Rayleigh, Sec.R.S.; Treasurer, A. B. Kempe, F.R.S.; Hon. Secs., M. Jenkins, M.A., R. Tucker, M.A.; other Members, Prof. Cayley, F.R.S., E. B. Elliott, M.A., Prof. Greenhill, M.A., J. Hammond, M.A., Prof. Hill, M.A., C. Leudesdorf, M.A., Captain P. A. Macmahon, R.A., S. Roberts, F.R.S., and J. J. Walker, F.R.S.

The new President, having taken the chair, thanked the members present and the Society for the honour they had done him, and then called upon Mr. Glaisher to read his Presidential Address. After the reading, Mr. S. Roberts moved that the Address be printed in the Society's *Proceedings;* the motion, having been seconded by Mr. Greenhill, was carried by acclamation.

The following communications were then made:—

Certain Operators in connection with Symmetric Functions: Mr. Lachlan.

On the Transformations of the General Elliptic Element $\frac{\delta x}{\sqrt{U_x}}$, where $U_x = x-\alpha.x-\beta.x-\gamma.x-\delta\,(=ax^4+4bx^3+6cx^2+4dx+e)$: Mr. Robert Russell.

Discussion of a Multilinear Operator, with applications to the Theories of Invariants and Reciprocants: Captain Macmahon, R.A.

The Theory of Screws in Elliptic Space (Fourth Note): Mr. Buchheim.

The Rectification of certain Curves: Mr. R. A. Roberts.

The Rectification of a Sphero-Conic: Mr. H. F. Burstall.

Third Paper on Reciprocants: Mr. L. J. Rogers.

The "Sine-Triple-Angle" Circle: Mr. Tucker.

The following presents were received:—

"Educational Times," for November.

"A Synopsis of Elementary Results in Pure Mathematics," by G. S. Carr, M.A. royal 8vo; London, 1886: from the Author.

"Beiblätter zu den Annalen der Physik und Chemie," Band x., St. 10.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. vii., No. 2; Coimbra, 1886.

"Bollettino delle Publicazioni Italiane," ricevute per Diritto di Stampa, Nos. 19, 20; Firenze.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. ii., Fasc. 6 and 7; Roma.

"Memorie della R. Accademia di Scienze, Lettere, ed Arti," in Modena, Serie II., Vol. III., 4to; Modena, 1885.
"Rendiconti dell' Accademia delle Scienze, Fisiche, e Matematiche," Napoli, Anno XXII., 1883; XXIII., 1884; XXIV., 1885; and XXV., 1886; (Fasc. 1, 2, and 3), 4to; Naples.

Presidential Address.

By J. W. L. Glaisher, M.A., F.R.S.

The Mathematical Tripos.

It is with the greatest pleasure that I avail myself this evening of the already well-established custom which permits one of our members, once in two years, to address to his colleagues a few general remarks connected with the science that forms our common bond of union. It is not often that a mathematician has an opportunity of laying before his fellow-workers, by word of mouth, any views of his own, except such as relate to the actual mathematical investigations upon which he is engaged, which, from their very nature, can appeal directly only to the few who have laboured in the same field; and I feel it to be a high privilege to be permitted, in this room, and surrounded by familiar faces, to give expression to my thoughts and hopes upon subjects that are of common interest to us all as mathematicians.

I have not ventured to attempt any remarks upon the wide region of pure mathematics, or even upon the progress of such portions of it as have attracted the greatest share of interest among ourselves. I have felt that, as one who has resided and lectured in Cambridge for the past fifteen years, the most appropriate subjects for my address would be those upon which my residence in the University during an eventful period, or my experience as a lecturer, might to some extent qualify me to speak. Still, even when so restricted, I have found it no easy matter to decide upon the subjects to which I was most desirous of drawing your attention to-night.

I should have liked to speak at length upon the theory of elliptic functions. For fourteen years I have lectured regularly, each year, upon this subject, and no lectures of mine have been of so much in-

terest to me. I believe that the time is rapidly approaching when the elementary portions of the theory will be regarded as necessarily forming a part of the common course of reading of all students of mathematics, so that a familiarity with sn's, cn's, dn's, and their properties, will become as essential as the differential calculus to the mental analytical equipment of every person who has made mathematics one of his subjects of study.

Quite apart from its far-reaching influence in all branches of pure mathematics, and its widespread applications in mathematical physics, there are special reasons which make the theory of elliptic functions a subject of peculiar interest in a course of mathematical studies, and one to which it is important that the student should be introduced as early as possible in his career, whether he be reading mathematics for its own sake, or for the sake of its applications, or for its advantages as a mental training. It is the first mathematical "theory" that he meets with in his reading — meaning by a "theory" a body of theorems and properties of functions so related to each other that the student cannot fail to see from the equations themselves that they form a consistent and remarkable system of facts, worthy of study on their own account, irrespective of any applications of which they may be susceptible. It is true that trigonometry, if regarded as the theory of singly periodic functions, is a theory in this sense, but it is reached by the student at too early a stage for him to be enabled to appreciate the nature and importance of facts that are expressed in the mathematical language of formulæ; and, even if it were not so, the manner in which the subject is treated in text-books (the functions being derived from the circle and applied to the solution of triangles, &c., before they are considered analytically) makes it difficult to separate the mathematical theory from its various applications.

In analytical geometry, which the student next meets with in his reading, a method of representing curves by equations is explained, and applied to the investigation and proof of properties of conics. In his next subject, differential calculus, he is introduced to new conceptions and processes of the very highest importance and the most fundamental character, and is taught to apply them to the investigation of maxima and minima, tangents and asymptotes to curves, envelopes, &c. Then come the elements of the integral calculus and of differential equations; the former consisting of a few chapters giving methods of integrating various classes of functions, followed by applications to curves and surfaces, and the latter of rules and methods for treating such equations as admit of finite solution. Not one of these subjects, in the form in which they are necessarily presented to students, is an end in itself or exists for itself: they con-

sist of ideas, methods, processes, and rules, which the student is taught to apply and to understand; they contain the conceptions with which he has to make himself as familiar as with the commonest facts of life, the tools which he is to have ever ready to his hand for use. In the course of acquiring this knowledge, he is made acquainted with numerous connected series of propositions—such as the properties of conics—besides various important results of more purely analytical interest. But all of these developments are presented to him in a form which throws no light upon the manner in which they were originally discovered, and, though the propositions are made to follow one another in clear logical order, the student cannot but be sensible that he is travelling, not along a natural highway, but upon a well-worn road, artificially constructed for his convenience.

It is not till he reaches the subject of elliptic functions that he has the opportunity of seeing how, by means of the principles and processes that he has learned, a theory can be developed, in which one result leads on of itself to another, in which every system of formulæ suggests ideas and enquiries about which the mind is eager to satisfy itself and opens to the view fresh formulæ connected by unsuspected relations with others already obtained, so that he cannot resist the feeling that the subject is taking its own course, and that he is merely a bewildered spectator, delighted with the results which unfold themselves before him. He feels that the formulæ are, as it were, developing the subject of themselves, and that his part is passive: it is for him to follow where the formulæ point the way, and to be amazed by the new wonders to which they lead him.

It may be that in using this language I am expressing the feelings of a mathematician, rather than those of a student on reading the elements of the subject for the first time; still, I am convinced that the attributes I have just referred to are those which distinguish a genuine mathematical theory from a mere collection of useful principles and facts, and that no one can have studied elliptic functions without realising that mathematics is not only a weapon of research, but a real living language—a language that can reveal wonderful and mysterious worlds of truths, of which, without its help, the mind could have gained not the least conception. It seems to me, therefore, of the highest importance that the student should be introduced to a real mathematical theory at the earliest stage at which his knowledge will permit of his deriving from it the peculiar advantages which I have mentioned. Thus only can he obtain expanded views or a true understanding of the science he is studying. Higher algebra and the theory of numbers afford other conspicuous examples of the perfection that a pure mathematical theory can exhibit, but they

do not lie so directly in the line of a general mathematical course of studies. Regarded from this latter point of view, elliptic functions has the additional merit of being a subject whose importance is recognised, on account of its physical applications, even by those to whom the gift of duly appreciating the wonders of pure mathematics seems to have been partially denied.

I should have liked also to speak at some length upon another subject that is constantly in my thoughts: I mean the pressing need of text-books upon the higher branches of mathematics. Of text-books for use in schools we have an abundance, and each month produces a fresh supply; but it is only occasionally that we have to welcome a work intended for the use of the higher University student or the mathematician. Every one of us must sometimes have felt the want of an introductory treatise that would give the reader the fundamental propositions in some branch of mathematics which exists only in memoirs and papers scattered throughout the wilderness of *Journals* and *Transactions* of Societies. We can scarcely expect to have provided for us, in many high subjects, text-books so admirable and thorough as Dr. Salmon's; still, I cannot refrain from expressing the hope that in the future the number of advanced mathematical treatises may not be so infinitesimal compared with the number of memoirs as at present. I could mention several subjects that are almost at a standstill, because advance is impracticable for want of avenues by which new workers can approach them. Of necessity the literature of mathematics must always be in the main a journal literature, for the audience addressed is small; but I cannot help feeling that the disproportion between the amount of exploration effected and the attempts made to render accessible the territories explored and conquered should be less than it is. No one can realise more vividly than I do how vastly more difficult it is to write a book than a collection of memoirs, and how beset with anxieties, for any one who is at all fastidious, is the task of arranging the fundamental properties of any comparatively new subject in clear and logical form. The sustained struggle to attain clearness, exactitude, and thoroughness in the orderly development of a complicated and mutually-connected system of propositions wears out the worker more than thrice the same amount of labour devoted to new investigations, with all the fascinating excitement of successes and failures, rewards and disappointments. In writing a memoir, the mathematician begins where he pleases, and confines himself to what has interested him and what he knows he has done well. In composing a book, the author has not only to marshal into order an array of theorems of various kinds, assigning to each its due place and importance, but he has—hardest

task of all, perhaps—to confine his treatise within bounds, to keep it from growing to gigantic proportions as his increased study of the subject opens up to him fresh vistas. On the other hand, however, is to be considered the great service he can thus render to his favourite study: an introductory treatise on a subject not otherwise approachable by any direct route, even if it be not of the highest class, may have done far more for its advance than could have been effected by the most brilliant memoir. Time, care, and thought are essential for the preparation of any valuable treatise, and full references to the original memoirs should be always given; if these conditions have been fulfilled, the writer has deserved well of mathematical science.

I have not been able to refrain from making the few preceding remarks upon two subjects on which I have long felt strongly; but I pass now, without further delay, to the main subject of my address —the Mathematical Tripos. I have thought that, in view of the importance of this examination to our science, and the frequent changes that have taken place recently, this might be a subject of no ordinary interest to our members as well as to myself. Since 1872 change has succeeded change with great rapidity, and there are probably not many, outside the mathematical portion of the resident body at Cambridge, who are fully aware of the present mode of conducting the examination, or of the further changes already sanctioned by the Senate and which take effect next June. It is, indeed, generally known that the list of wranglers, senior optimes, and junior optimes, is now published in June, at about the same time as many other Tripos lists, instead of by itself in January, and that the senior wrangler is displaced from his throne, and no longer owes his position to the results of the whole examination, so that he is not necessarily —even from an examination standpoint—the first mathematician of his year. So much only is generally known; and it has seemed to me that it might be of interest, considering the influence for good that it is hoped the examination in its new form will have upon the progress of mathematics, to give some account of the successive developments that have taken place in this time-honoured examination, and of the causes and efforts that have led to them. The difficulties connected with the placing of all the mathematical candidates of the year in one order of merit, the extension or limitation of the subjects of examination, and various other questions connected with the Tripos, are matters that have been continually discussed and re-discussed in the light of fresh experience by those concerned with the mathematical course of studies at Cambridge; but I may, nevertheless, perhaps be permitted to-night very briefly to refer to some of the

familiar arguments in the presence of a more extended audience of mathematicians.

It is convenient to preface the principal remarks I have to make by an outline of the history of the Tripos. In doing so, I must pass very lightly over its origin and early development; for anything approaching to a complete history of its origin and rise in the last century would amount almost to a history of the studies of the University.

At the beginning of the last century, except certain merely formal disputations, the only exercises required from candidates for degrees were the keeping of acts and opponencies. Each candidate for honours, in the course of his third year, had to maintain publicly a thesis, the subject of which was chosen by himself, against three opponents, in the presence of one of the Moderators who acted as umpire. The subjects selected were philosophical or mathematical; the discussion took place in Latin, and in logical form. After hearing the discussions, the Proctors and Moderators prepared a final list of candidates qualified to receive degrees. This can scarcely be considered to have been an order of merit, for the Vice-Chancellor, the Proctors and the Senior Regent had the right to introduce four additional names into any part of the list; still, except in the case of the recipients of these honorary degrees (called proctors' optimes), it is probable that the list in the main fairly represented the merits of the candidates. It was divided into three classes, consisting of (i.) the wranglers and senior optimes, (ii.) the senior optimes who had done fairly well but had not distinguished themselves, and (iii.) οἱ πολλοί, or the poll-men. The first class received their degrees on Ash Wednesday, taking seniority according to their order on the list, and the two other classes received their degrees later.

With regard to the origin of the Tripos, Mr. W. W. Rouse Ball, in an interesting sketch of its history,* writes:—

"The impressions gathered from these disputations in the schools were necessarily rather vague, and when they became the sole University exercise for a degree they hardly afforded a sufficient basis for an accurate arrangement of the men in order of merit. It was, I believe, to correct this fault that the Senate House examination was introduced, and I am inclined to think that it had its origin about the year 1730. At first it probably consisted only of a few *vivâ voce* questions addressed by the Proctors and Moderators in the week after the schools to those candidates about whose abilities and position some doubt was felt; but its advantages were so patent that within ten or twelve years it had become systematised into a regular examination to which all questionists were liable, although technically it was still regarded as only supplementary to the exercises in the schools. From the beginning

* "The Origin and History of the Mathematical Tripos," Cambridge, 1880. (Reprinted from the *Cambridge Review*.)

it was conducted in English, and accurate lists were made of the order of merit of the candidates; two advantages to which, I think, its final and definite establishment must be largely attributed."

Mr. Ball divides the time during which the exercises in the schools and the Tripos were concurrent into five periods: (1) from 1730 to about 1750, during which time it was probably unauthorised, and regarded as an experiment; (2) from 1750 to 1763, during which it was gradually establishing itself,—in the last year of this period it was officially decided that, when a candidate's position in the class-list was doubtful, the Senate House examination, and not the disputation, was to be taken as the final test; (3) from 1763 to 1779, during which definite rules were framed and laid down for conducting it; (4) from 1779 to 1827, during which it practically superseded the disputations; (5) from 1827 to 1839, the year in which the disputations were discontinued.

The lists published in the Cambridge University Calendars begin with the year 1747, because in that year the final lists were first printed and distributed, the names of those who had received honorary degrees being specially marked, so that by simply erasing them the true order of merit of the other candidates could be obtained. The division of the first class into wranglers and senior optimes was first made in 1753.

It was in the third of the above periods, that is, between 1763 and 1779, that the Senate House examination was gradually gaining ground upon the schools in determining a candidate's final place on the list. By means of their acts and opponencies the candidates were divided by the Moderators into eight classes, each class being arranged in alphabetical order; their subsequent position in the class was then determined by the Senate House examination. The first two classes comprised those who were expected to be wranglers, the next four included the other candidates for honours, and the last two consisted of poll-men only. The classes were examined separately and *vivâ voce*. During this period it became the custom to require written answers to the questions. The examiner gave out the questions to the class one by one, giving out a fresh question as soon as he saw that any one had finished the last. The problem papers, which were confined to the first two classes, were given to the candidates in writing, so that they had the whole paper before them at once.

It may be of interest to give a more detailed account of the exercises in the schools during this period, when both the exercises and the examination were in full operation and vigour. The Moderators, having received from the tutors of the Colleges a list of the students who were candidates for honours at the next examination, fixed a day

in the Lent term on which each was to keep his act, and assigned to him three opponents. The Respondent, or "Act" as he was then called, selected three subjects which he proposed to maintain, and submitted them to the Moderator, who communicated them to his three opponents, designating them *opponentium primus*, *secundus*, or *tertius*. On the day fixed for the Act the respondent read his thesis in the schools, in the presence of the Moderator. The first opponent then mounted the box opposite to that of the respondent and below that of the Moderator, and joined issue with him, opposing the thesis by eight arguments of syllogistic form. The respondent replied to each in turn, and, when an argument had been disposed of, the Moderator called for the next in the words *Probes aliter*. When the disputation had continued long enough, the Moderator dismissed the opponent with such words as "*Bene disputasti*," or "*Optime disputasti*," or "*Optime quidem disputasti*," as the case might be. The second and third opponents (who had to oppose the thesis by five and three arguments respectively) entered the box successively, and after disputing were dismissed in the same manner, the whole performance lasting between one hour and two hours. The respondent himself was dismissed with some such phrase as "*Satis et optime quidem tuo officio functus es.*" Such compliments gave rise to the classification into senior and junior optimes. In general, "*Optime quidem*" was the highest praise expected even by future wranglers. The distinguished men of the year appeared eight times in the schools, twice as Respondents, and twice in each grade of opponency.*

The increasing importance of the examination is shown by the regulations agreed to by the Senate in 1779. The Moderators of the previous year were then added to the regular staff of examiners, and the system of brackets ("classes quam minimæ") was

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* Wordsworth, *Scholæ Academicæ* (1877), p. 37. A specimen of an argument, expressed in scholastic form, on the question, "Recte statuit Paleius de Virtute," is given by Wordsworth on p. 39, and the full system of eight arguments (in a disputation of 1784) on the question, "Solis parallaxis ope Veneris intra solem conspiciendæ a methodo Halleii recte determinari potest," is reproduced in detail by Mr. Ball in the appendix to the sketch already referred to. In the latter part of the last century, it seems to have been usual for two of the questions to relate to mathematics, and the third to moral philosophy. Wordsworth mentions that in 1710—11 it needed all the influence of an enthusiastic Proctor and Moderator to induce a student to keep his act in mathematical questions, but that by the middle of the century the examination was so far crystallising into the *Mathematical* Tripos, that a questionist was enabled by academical authority in 1750 to resist the demands of a Moderator to produce one metaphysical question, he having already distinguished himself in mathematical argument. In the early Cambridge University Calendars the three questions given as specimens are, (1) "Recte statuit Newtonus in septima sua sectione Libri primi"; (2) "Iridis primariæ et secundariæ Phenomena solvi possunt ex Principiis Opticis"; (3) "Recte statuit Lockius de Qualitatibus Corporum."

introduced. The examination lasted three days (the last of which was devoted to moral philosophy), and on the fourth day a class-list, called "the Brackets," was issued, in which those candidates who were nearly equal were bracketed together. One day was devoted to the "examination of the brackets," by the result of which the names in each bracket were placed in order of merit. There was also a power of challenging, by which a candidate who was dissatisfied with his bracket might challenge any other candidate he pleased to a fresh examination*; but it seldom happened that any one rose above or fell below his bracket. From 1779 onwards the examination slowly and surely grew in importance, and the exercises became of less account each year, till they were finally discontinued in 1839.

The following account of the Senate House examination in 1802 is abridged from the Cambridge University Calendar of that year †:—

"On the Monday morning, a little before eight o'clock, the students, generally about a hundred, enter the Senate House, preceded by a Master of Arts, who on this occasion is styled the father of the College to which he belongs. On two pillars at the entrance of the Senate House are hung the Classes (*i.e.*, the eight classes into which the candidates have been divided by the exercises in the schools) and a paper denoting the hours of examination of those who are thought most competent to contend for Honours. Immediately after the University clock has struck eight, the names are called over, and the absentees, being marked, are subject to certain fines. The classes to be examined are called out, and proceed to their appointed tables, where they find pens, ink, and paper provided in great abundance. In this manner, with the utmost order and regularity, two-thirds of the young men are set to work within less than five minutes after the clock has struck eight. There are three chief tables, at which six examiners preside. At the first, the senior Moderator of the present year and the junior Moderator of the preceding year. At the second, the junior Moderator of the present and the senior Moderator of the preceding year. At the third, two Moderators of the year previous to the two last, or two examiners appointed by the Senate. The two first tables are chiefly allotted to the six first classes; the third, or largest, to the *οἱ πολλοί*."

After describing the manner of reading out the questions, the account proceeds:—

"The examiners are not seated, but keep moving round the tables, both to judge how matters proceed and to deliver their questions at proper intervals. The examination, which embraces arithmetic, algebra, fluxions, the doctrine of infinitesimals and increments, geometry, trigonometry, mechanics, hydrostatics, optics, and astronomy, in all their various gradations, is varied according to circumstances: no one can anticipate a question, for in the course of five minutes he may be dragged from Euclid to Newton, from the humble arithmetic of Bonnycastle to the abstruse analytics of Waring. While this examination is proceeding at the three tables between the hours of eight and nine, printed problems are delivered to each person

* In such cases the Moderators called to their assistance the Proctors or other Masters of Arts. Any Master of Arts was at liberty to examine any of the candidates.

† Dr. Jebb's detailed account of the exercises and examinations as they existed in 1772 (which was applicable with very little alteration till 1828) is reproduced in Whewell's *Of a Liberal Education* (1848). This work contains a discussion of the changes made in the examination between 1772 and 1845.

of the first and second classes; these he takes with him to any window he pleases, where there are pens, ink, and paper prepared for his operations."

At nine o'clock the papers had to be given up, and half an hour was allowed for breakfast. At 9.30 all returned, and were examined in the same way till eleven, when the Senate House was again cleared. An interval of two hours then took place. At one o'clock all returned again and were examined. At three o'clock the Senate House was again cleared for half an hour, and, on the return of the candidates, the examination was continued till five o'clock. This ended the Senate House examination for the day, but at seven in the evening the first four classes went to the senior Moderator's rooms to solve problems. They were finally dismissed for the day at nine, after eight hours of examination.* The work on the next day (Tuesday) was similar to that of the Monday; Wednesday was devoted to logic, moral philosophy, &c. "On Thursday the examinations are resumed, and continued nearly as usual, as on the Monday and Tuesday. At eight o'clock the new classifications, or brackets, which are arranged according to the order of merit, each containing the names of the candidates placed alphabetically, are hung upon the pillars." Fresh editions and revisions of the brackets were published at 9 and 11 a.m., and at 3 and 5 p.m., according to the course of the examination, liberty being given to any candidate to challenge the bracket immediately above his own. At five the whole examination ended. The Proctors, Moderators, and examiners then retired to a room under the public library to prepare the list of Honours, which was sometimes settled without much difficulty in a few hours, but sometimes not before two or three the next morning. The name of the senior wrangler was generally published at midnight. In 1802, there were eighty-six candidates for honours, and they were divided into fifteen brackets, the first and second brackets containing each one name only, and the third bracket four names.

The examination seems to have been a time of great excitement, and the list was most anxiously awaited. This was the case also before 1779, as appears from the account of the contest between Paley and Frere for the senior wranglership in 1763, and from Jebb's account of the examination in 1772.†

Various changes took place in the examination in 1808, 1828, 1833, and 1839. In 1808 another day was added: three days were devoted

* In 1818 the hours for the evening problem paper were 6—10, so that the candidates had ten hours of examination in the day. Originally, as mentioned above, the problems were only set to the first two classes; in 1802 they were open to four classes, and in 1818 to all six classes, *i.e.*, to all the candidates for honours.

† Wordsworth (pp. 47 *et seq.*). See also the letters of Gooch, who was second wrangler in 1771 (p. 322).

to mathematics, exclusive of the day of examination of the brackets. Each candidate was employed eighteen hours in the course of the three days, of which eleven were devoted to book-work and the remaining seven to problems. By regulations which were confirmed by the Senate on November 13, 1827, and came into operation in January 1828, another day was added, so that the examination extended over four days, exclusive of the day of examining the brackets; the number of hours of examination was twenty-three, the time assigned to problems being the same as in 1808. On the first two days all the candidates had the same questions proposed to them, inclusive of the evening problems, and the examination on those days excluded the higher and more difficult parts of mathematics, in order, in the words of the report, "that the candidates for Honours may not be induced to pursue the more abstruse and profound mathematics, to the neglect of more elementary knowledge." Accordingly, only such questions as could be solved without the aid of the differential calculus were set on the first day, and those set on the second day involved only its elementary applications. The classes were reduced to four, determined as before by the exercises in the schools. The regulations of 1827 are especially important because they first prescribed that all the papers should be printed.* They are also noticeable as being the last which gave the examiners power to ask *vivâ voce* questions. After recommending that there be not contained in any paper more questions than well-prepared students have generally been found able to answer within the time allowed for the paper, the report proceeds:—"But, if any candidate shall, before the end of the time, have answered all the questions in the paper, the examiners may at their discretion propose additional questions *vivâ voce*."

New regulations were confirmed by the Senate on April 6, 1832, and took effect in 1833. The commencement of the examination was placed a day earlier, the duration was five days, and the number of

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* The words of the report are:—"It is further recommended that the questions from books, which have hitherto been proposed to the classes *vivâ voce*, should, for the future, be printed. And it is hoped that, as by this means the questions proposed in each year will be more generally known, the students may thus be better directed in their reading than they now are, and the mathematical studies of the University become more fixed and definite. An opportunity will also be afforded of ascertaining by an inspection of these papers that the examination embraces a due proportion of each of the ordinary subjects of mathematical study.

"As, however, in proposing this alteration, the intention of the Syndicate is to avoid making any change in the substance of the examination, it is recommended that the examiners be strictly enjoined to insert in these papers such questions only as are at present proposed *vivâ voce;* namely, propositions contained in the mathematical works commonly in use in the University, or simple examples and explanations of such propositions."

hours of examination on each day was five and a half. Thus four and a half hours were added to the whole time of examination, four of which were assigned to book-work, the remaining half-hour being added to the time devoted to problems. The examination on the first day was confined to subjects that did not require the differential calculus, and on the second and third days only the simple applications of the calculus were included. During the first four days of the examination the papers were set to all the candidates alike, but on the fifth day the examination was conducted according to classes. No reference is made to *vivâ voce* questions, and the examination of the brackets only survives in the permissive form:—"That the result of the examination be published in the Senate House on the morning of the Friday succeeding the first Monday in Lent term, at nine o'clock; but, if it should happen that the relative merits of any of the candidates are not then determined to the satisfaction of the Moderators and Examiners, that such candidates be re-examined on that day."

Only six years later these regulations were superseded by a new system, which passed the Senate on June 2, 1838, and came into operation in January 1839. By these new regulations, another day was added to the examination, which thus lasted six days. The total number of hours of examination was thirty-three, of which eight and a half were given to problems. Throughout the whole examination the same papers were set to all the candidates. The regulations contain no mention of classes, and accordingly the exercises in the schools were discontinued by the Moderators. The permissive rule relating to the re-examination of the candidates (the survival from the brackets) was retained in these regulations in the same form as in those of 1832.

It is very interesting to notice, in the successive regulations, how the *vivâ voce* examination gradually merged into an examination by printed papers, and how, as the examination became more elaborate and exacting, it rendered unnecessary, not only the preliminary exercises in the schools, but also the final examination of the brackets.

Besides the development of the Senate House examination itself, other changes had taken place in the University system during this period of renewed activity. In 1824 the first Classical Tripos examination took place; only those who had already passed the mathematical examination being admissible as candidates. The name "Mathematical Tripos"* dates from after this year. Previously it

* There are few matters in the history of the University more curious and interesting than the manner in which the word *Tripos* came to be applied to the Senate House examination, and consequently also to the other final honour examinations. It is natural to suppose that it is connected with the three classes into which the

had been known as the Senate House examination, and this name continued long afterwards, and for more than thirty years was still printed as a heading to the papers set. It was only as a means of distinguishing it from the Classical and other newer Triposes that the name Mathematical Tripos gradually came into use. By the regulations which took effect in 1828 a totally distinct series of papers were set to the poll-men, who then formed the fifth and sixth classes. The fiction of regarding the poll-list as a continuation of the list of mathematical honours still lingered till 1858, the names being arranged in order of merit in four classes, called the fourth, fifth, sixth, and seventh; the fourth class being in theory supposed to be the next class to that of the junior optimes.

Very important regulations came into effect in 1848. The exami-

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honour list was divided, but there is no connection whatever. The history of the name may be given briefly as follows:—In the ceremonies which were performed on Ash Wednesday in the middle of the sixteenth century, at the admission of questionists to be Bachelors of Arts, an important function was executed by a certain "ould bachilour," who was appointed as champion on the side of the University. He had to "sit upon a stoole before Mr. Proctours" and to dispute with the "eldest son" (the foremost of the questionists), and afterwards with "the father" (a graduate of the College to which the "eldest son" belonged, representing the paternal character of the College). At this time the only "Tripos" was the three-legged stool on which the Bachelor sat. A century later this Bachelor seems to have become a sort of licensed buffoon, and to have been called "Mr. Tripos," just as a president is sometimes referred to as "the Chair," or a judge as "the Bench." During the 120 years in which the name Tripos or Tripus indicated a personage, there are frequent allusions to the humorous orations delivered in the schools by those who filled this office. These were known as Tripos speeches. It is probable that Mr. Tripos ceased to take part in the arguments in the schools between 1730 and 1750. The Tripos speeches were discontinued, but the Latin verses, written by the two Messrs. Tripos of the year, and circulated on the admission days, were still retained. These were called Tripos verses. About 1747 the Moderators began the custom of printing Honour lists on the back of the Tripos verses. Thus the list of Honours in the Senate House examination came to be called the Tripos list, so that a man's name was said to stand in such a place in the Tripos of his year, *i.e.*, upon the back of the Tripos verses. And, lastly, the name was transferred from the list to the examination, the results of which were published in the list. See Wordsworth, *Scholæ Academicæ*, Chap. ii.: he concludes his account: "Thus, step by step, we have traced the word *Tripos*, passing in signification, Proteus-like, from a thing of wood (*olim truncus*) to a man, from a man to a speech, from a speech to two sets of verses, from verses to a sheet of coarse foolscap paper, from a paper to a list of names, and from a list of names to a system of examination."

The earliest Tripos verses in the Registry of the University are those of the year 1575. The Tripos verses have never been discontinued. At present they are published on the last day of the Michaelmas term, and consist of four sets of verses, usually in Latin, but occasionally in Greek, in which current events or topics of conversation in the University are treated satirically or seriously. They are written for the two Proctors and two Moderators by undergraduates or commencing bachelors, and are printed on the two inside pages, or on the first three pages, of a sheet of foolscap paper. On the fourth page the order of seniority of the honour men of the year is printed crosswise in columns, the sheet being folded into four parts so that all the names can be read without opening the page to more than half its extent.

nation, as thus constituted, underwent no further alteration till 1873, and the first three days remain practically unchanged at the present time. The duration of the examination was extended from six to eight days, the first three days being assigned to the elementary, and the last five to the higher parts of mathematics. After the first three days, there was an interval of eight days (soon afterwards increased to ten), and at the end of this interval the Moderators and Examiners issued a list of those who had so acquitted themselves as to deserve mathematical honours. Only those whose names were contained in this list were admitted to the five days, and, after the conclusion of the examination, the Moderators and Examiners, taking into account the whole eight days, brought out the list arranged in order of merit. No provision was made for any further examination corresponding to the examination of the Brackets, which, though forming part of the previous scheme, had been discontinued for some time. A very important part of the scheme was the limitation, by a schedule, of the subjects of examination in the first three days, and of the manner in which the questions were to be answered; the methods of analytical geometry and differential calculus being excluded. In all the subjects contained in this schedule, examples and questions arising directly out of the propositions were to be introduced into the papers, in addition to the propositions themselves. Taking the whole eight days, the examination lasted $44\frac{1}{2}$ hours, 12 hours of which were devoted to problems.

In the year in which these regulations came into force, the Board of Mathematical Studies (consisting of the mathematical Professors and the Moderators and Examiners for the current and the two preceding years) was constituted by the Senate. Although the new regulations had so strictly limited the subjects which could be set in the first three days, they had imposed no limitation whatever upon those which could be set in the last five days, the subjects of examination appearing in the schedule simply as pure mathematics and natural philosophy. Accordingly, the first matter to which the newly-constituted Board turned its attention was that of restricting the subjects on which questions should be set in the last five days of the examination.

It becomes necessary, therefore, at this point, to refer briefly to the range of subjects which were included in the examination. Of the nature of the questions proposed prior to 1828, the first year in which all the papers were printed, very little can be known except what can be gathered from the problem papers and the specimens of the other papers that have been preserved;* but there can be no doubt that

* The problem papers were, I believe, printed from 1779; but only those of the present century are accessible in the Cambridge University Calendars and other publications.

their character was determined by the ordinary Cambridge treatises then in use, which, it is well known, were far behind the corresponding treatises published on the Continent. Woodhouse's "Principles of Analytical Calculation" (1803), and "Plane and Spherical Trigonometry" (1809), are the earliest indications of the introduction of the analytical element into the mathematics of the University; a more decided impulse in this direction was given by the translation of Lacroix's "Differential and Integral Calculus," by Herschel, Peacock, and Babbage (1817), followed by Peacock's "Examples on the Differential and Integral Calculus," and Herschel's "Examples on the Calculus of Finite Differences" (1820).

The reform in the mathematical studies of the University which was effected by Herschel, Peacock, and Babbage, is well known. It is to them that we mainly owe the revival of mathematics in this country and the restoration of intercourse with the rest of Europe after three-quarters of a century of isolation. Peacock was Moderator in 1817, and he ventured to introduce the symbol of differentiation into the examination, his colleague, however, retaining the old fluxional notation. The old system made its appearance once more in 1818; but in 1819 Peacock was again Moderator, and, having a colleague who shared his views, the change was fully accomplished.

The introduction of the notation and language of the differential calculus into the Senate House examination forms an important landmark in the history of Cambridge mathematics. From that time onward the University began to make up slowly but surely the ground she had lost; step by step the analytical processes and methods superseded the older geometrical modes of treatment, and each year saw a substantial increase in the range of subjects included in the course of study.

Second only in importance to the revolution effected by the substitution of the differential for the fluxional calculus, was the rise of analytical geometry in the first thirty years of the century; and, considering the amount of attention that this subject has received at Cambridge in the last fifty years, and the accessions that have been made in this country to the analytical theory of curves and surfaces, a peculiar interest attaches to the introduction into the University of the algebraic treatment of geometry and to the early stages of its development. The first edition of Wood's "Algebra," which appeared in 1795, contained, as Part IV., a chapter of thirty pages "On the Application of Algebra to Geometry," in which are given the equations of the straight line, ellipse, cissoid, conchoid, and other curves, the construction of equations, &c. This chapter remained unchanged

in the ninth edition (1830), and seems to have formed the only introduction to analytical geometry existing in the University until 1826, when Hamilton* published his "Principles of Analytical Geometry, designed for the use of Students in the University." This was not the first English treatise on analytical geometry, for Lardner's "Algebraic Geometry" was published, three years earlier, in 1823; but it was the first Cambridge book, and the first which included solid geometry. The problem papers from 1800 to 1820 show that at the beginning of the century analytical geometry was always represented to some extent, though scarcely as an independent subject, most of the questions relating to areas, loci, &c., in which but little more than the mode of representation by means of ordinates and abscissæ was involved. Hymers published his "Analytical Geometry of Three Dimensions" in 1830, and his "Conic Sections" in 1837. The latter at once superseded Hamilton's treatise, and remained the standard work on the subject for many years.

In applied mathematics, the character of the questions proposed was largely influenced by the publication of Whewell's "Mechanics" (1819), Whewell's "Dynamics" (1823), Coddington's "Optics" (1823), Woodhouse's "Plane Astronomy" (1821—23), and Airy's "Tracts" (1826). A second edition of this last work, which appeared in 1831, contained a tract on the "Undulatory Theory of Light," a subject which was freely represented in the examination for many years. Not only were the questions modified, in character and range, by the publication of new mathematical treatises in the University, but they were also affected to a certain extent by some of the professorial lectures. At this time, too, the Smith's Prize examination exerted a beneficial effect upon the Senate House examination, certain classes of questions which were originally introduced into the former having shortly afterwards been admitted into the latter. Between 1830 and 1840, questions in definite integrals, Laplace's coefficients, electricity, magnetism, and heat were also introduced. There were no regulations of any kind, and the responsibility of introducing innovations and alterations rested solely with the Moderators and Examiners. The uncertainty as to the subjects that the examination would embrace, and the want of any due notice of any extension of them, were found to be a serious inconvenience to the higher class of students; although, as has been already stated, the introduction of a new subject had been generally preceded by the publication of a work, by a Cambridge mathematician, in which it was treated in a manner adapted to the examination.

* Late Dean of Salisbury; born April 3, 1794; died February 7, 1880.

The Board of Mathematical Studies was created by the Senate on October 31, 1848, and in May of the following year they issued a report to the Senate in which, after giving a short review of the past and existing state of mathematical studies in the University, they recommended that, considering the great number of subjects occupying the attention of the candidates, and the doubt existing as to the range of subjects from which questions might be proposed, the mathematical theories of electricity, magnetism, and heat should not be admitted as subjects of examination. In the following year they issued a second report, in which they recommended the omission of elliptic integrals, Laplace's coefficients, capillary attraction, the figure of the earth considered as heterogeneous, &c., besides certain limitations of the questions in lunar and planetary theory, &c. In making these recommendations, the Board expressed their opinion that they were only giving definite form to what had become the practice in the examination, and were only putting before the candidates such results as they might themselves have deduced by the study of the Senate House papers of the last few years. The Board also recommended that the papers containing book-work and riders should be shortened.

From 1823 onwards, the examination was conducted in each year by four examiners—the two Moderators and the two Examiners, the Moderators of one year becoming as a matter of course the Examiners of the next. Thus, of the four examiners in each year, two had taken part in the examination of the previous year. The continuity of the examination was well kept up by this arrangement; but perhaps it had the effect of causing its traditions to be rather too punctiliously observed, the papers of each year being, as regards the subjects included, exact counterparts of the corresponding papers of the previous year. The resolutions of the Board in 1849–50 were not binding on the successive Moderators and Examiners up to 1872, but each year they seem to have felt themselves bound to follow the precedent of their predecessors, so that no new subjects were introduced. One would suppose from an examination of the papers set that those of the last five days must have been framed in accordance with a schedule as precise and detailed as that which governed the first three days.

In 1865 the Board recommended that after 1866 Laplace's coefficients and the figure of the earth, considered as heterogeneous, should be included in the examination, but this appears to be the only extension of the range of subjects recommended by the Board during the time in which the regulations of 1848 remained in force.

The period that followed the constitution of the Mathematical Board was one of activity in the whole University. The first examinations of the Moral Sciences Tripos and of the Natural Sciences Tripos were held in 1851. In 1850 a Royal Commission was issued to inquire into the University and Colleges, and in 1852 their report was presented to Parliament. In consequence of this report, a bill was introduced into Parliament, which received the royal assent in 1856; and under its powers new statutes were framed, both for the University and for the Colleges. Amid all these changes the Mathematical Board, though not very active, was not idle. The subjects which chiefly occupied its attention were the alteration of the date of the first three days from January to the previous June (for, by recent changes, the poll-men were examined in June, and so received their degrees seven months before the mathematical honour men), and the introduction of the *vivâ voce* element into the examination. Neither of these innovations, though frequently discussed and finally recommended by the Board, was received with much favour in the University. With regard to the latter, the opinion seems now to have become general that an admixture of the *vivâ voce* element, however valuable it may be in the lecture-room, is useless, or even worse, in testing the proficiency of candidates with a view to arranging them in strict order of merit. The change of time from January to June was at length effected, as will be seen by the regulations which came into operation in 1882.

In 1866 the attention of the Board was directed to the exclusion of certain important branches of mathematics from the studies of the University, owing to the fact that they were not represented in the Tripos examination. The rewards attending a high place in the Tripos were so great that the reading of most of the best men was directed almost wholly to this end; it was, therefore, practically impossible to introduce new mathematical subjects into the University without assigning to them some place in the Tripos. Now, although the recommendations of 1849–50 had curtailed the range of subjects, the course had nevertheless extended itself in some directions—where the name of the subject permitted of such extension—and especially in analytical geometry and higher algebra. The fact that this extension was taking place in certain subjects, while others were wholly omitted, alone sufficed to show the need of some revision of the limitations imposed upon the subjects that might be set. The Board, after careful consideration, came to the conclusion that the time had come when it was desirable to allow the candidates a certain option with respect to the higher branches of mathematics, and that this could be effected by increasing the number of subjects, and arranging

them in several divisions, over which the marks were distributed in a known proportion. Each candidate would be at liberty to devote himself to such of the divisions as he thought most advantageous, there being nothing to prevent his taking up all the divisions, if it were possible for him to do so. In a report dated May 8, 1867, the Board gave expression to these views, and recommended a scheme for the five days, according to which the subjects of examination were arranged in five divisions, with an approximately determinate number of marks assigned to each division. The subjects included in the five divisions were thirty-five in number, and included elliptic integrals, elastic solids, heat, electricity, and magnetism. On June 3, 1867, a syndicate was appointed by the Senate to consider the proposals of the Board; and the regulations recommended by this syndicate were approved by the Senate on June 2, 1868, and came into operation in January 1873.

In this new scheme of examination the three days were left unchanged, and the schedule of subjects for the five days, and their arrangement in divisions as proposed by the Board, were adopted with very slight modifications, the marks awarded to the five divisions being to those awarded to the three days in the proportion of 2, 1, 1, 1, $\frac{2}{3}$ to 1 respectively.*

The new regulations also made two other changes of importance: they added an extra day to the examination, and increased the number of the examiners from four to five. The extra day was the day immediately following the three days, and it was devoted to easy questions upon the subjects in the five days' schedule. Although the papers set on this fourth day were put before all the candidates, they were taken into account along with the five-day papers, and not with the three-day papers; so that this day had no effect upon the alphabetical list of those who deserved mathematical honours; which, as before, was dependent upon the three-days' marks only.

The Additional Examiner was appointed on the nomination of the Mathematical Board, and held office for one year only; and, to render his duties as little irksome as possible, he was not required to take part in the first three days—the most laborious part of the examination, so far as the looking over the papers is concerned, on account of the quantity of work sent in. It was thought that, in introducing the new subjects of electricity and magnetism into the examination, certain non-resident Cambridge mathematicians, whose names were

* The regulation assigning the proportion of marks to be awarded to the different divisions, was one which was found in practice very difficult to carry out, even approximately.

closely connected with great recent advances in these subjects, might be willing to give the University the benefit of their assistance, and that the influence of eminent non-resident mathematicians upon the examination, and therefore also upon the course of studies in the University, would be of the greatest value. These hopes were abundantly justified.

The general working of the new system soon disclosed the fact that the desired effect of inducing the best candidates to make a selection from the higher subjects, and to concentrate their reading, had not been attained. It was found that, unless the questions were made extremely difficult, more marks could be obtained by reading superficially all the subjects in the five divisions than by attaining real proficiency in a few of the higher ones; and the best men of the year were tempted, not to say compelled, to extend their reading as widely as possible over the book-work of the whole range of subjects. Thus, with respect to the main object which the framers of the scheme had in view, it was a complete failure.

Accordingly, on May 17, 1877, a syndicate was appointed by the Senate to consider the higher mathematical studies and examinations of the University. This syndicate consisted of eighteen members, representing nearly all phases of mathematical research and study in Cambridge; they met every week during the whole academical year, and the thorough examination and discussion that the subject received, both on the syndicate and in the University at large, brought out in the strongest light how great were the intrinsic difficulties connected with the retention of the order of merit, and how wide was the diversity of opinion—so much so, that at one time it seemed almost hopeless to attempt to devise a scheme that should receive a fair amount of general support. Even when the subjects were restricted, as they had been in the twenty-five years from 1848–72, it was sufficiently difficult to include in one list all the various classes of candidates—those who may be described as professed mathematicians, who intended to devote themselves to mathematics after their degree as investigators or teachers; those who adopted mathematics as their subject of study on account of its unrivalled mental training, and subordinated their whole reading to the single object of obtaining the highest place their abilities would enable them to reach; and those who, without any hope of obtaining a good place in the list, desired, nevertheless, to graduate in the Tripos on account of the high position held by mathematics among the branches of a liberal education. But, when the range of subjects was extended, there was the further dilemma: if there was to be a single order of merit, all the questions must be submitted to all the candidates; but, if a candidate

was to be at liberty to attempt all the questions, it appeared that, under any scheme that could be devised, the best candidates would find it more to their advantage to read the elementary portions of all the higher subjects than the higher portions of a few. If the questions were to be alternative, how could the order of merit be retained? How is it possible to compare one student's elliptic functions with another's elastic solids? was a question often asked.

It was keenly felt in the University that subjects like heat, electricity, and magnetism could not, with propriety, be omitted from the course systematically studied by candidates for mathematical honours. On the other hand, it was universally admitted that, by the extension of the range of subjects, the severe strain of the competition had been intensified to an injurious extent; and not only had the addition of the new subjects aggravated the evils arising from excessive competition, but they had even caused a deterioration in the quality of the work of many of the students, who were led, in the hope of gaining higher places, to attempt matter really beyond their grasp. The opinion was expressed on the syndicate that the only escape from the dilemma was by abandoning the order of merit. The majority, however, preferred to attempt some other remedy, without interfering with the final form of the Tripos list, which had been of such immense service to the University in the past, and was connected with so many valued associations. They therefore proposed, as the only method by which the pressure on the mathematical candidates could be relieved, to omit a varying portion of the higher subjects of examination in each year.

The report of the syndicate was presented to the Senate on March 29, 1878. They recommended that the nine days of the examinations should be divided into three groups of three days each, called Parts I., II., and III. Part I. was to be the same as the first three days in the schemes that came into operation in 1848 and 1873. Part II. was to be conducted according to a schedule of subjects considerably more restricted than the unwritten schedule that ruled the five days from 1848 to 1872. It included the more elementary portions of most of the ordinary subjects, such as differential equations, hydrostatics, rigid dynamics, optics, spherical astronomy, &c., but excluded calculus of variations, thermodynamics, physical optics, &c. It was proposed to move forward the time of examination in these first two parts, from January to the previous June, *i.e.*, two years and nine months from the time of coming into residence of the students. After the examination in Parts I. and II. a list of the candidates was to be published, arranged in three classes as before, the senior and junior optimes being placed in order of merit, and the wranglers in

alphabetical order. Only the wranglers were to be admitted to Part III., which was to take place at the old time in the following January. A final list was then to be issued, in which the wranglers were to be placed in order of merit, the marks obtained by them in all three parts being added together. The most important part of the scheme was the schedule of subjects for Part III. It contained all the subjects which were included in the schedule of the five days of the then existing examination (for the syndicate had decided that they would neither propose the addition of any new subjects nor the omission of any that had been already included), divided into three groups, A., B., C. It was recommended that questions from the subjects in group A. should be set every year, and that questions from groups B. and C. should be set in alternate years. It was thus proposed to establish, as it were, a "rotation of subjects." This scheme was voted upon by the Senate on May 13, 1878, when all its essential features were rejected. The division of the examination into three parts, of which the first two should take place in June and the third in the following January, was agreed to, as also was the limitation of Part III. to wranglers; but the carrying over the marks of the wranglers from June to January and the proposed rotation of subjects were rejected.

It is evident that the acceptance of the scheme, even by those who assented to its principle, depended largely upon the manner in which the subjects were divided into the three groups A., B., C. Whether a satisfactory division of the subjects was possible, is very doubtful; but it is certain that the grouping of the subjects proposed by the syndicate was extremely unsatisfactory.

This scheme, though it never came into operation, will be memorable in the history of the Tripos as the final attempt made to retain the order of merit in its old form. With its rejection there passed away all hope of expressing the results of the whole examination by means of a single order of merit. The scheme also deserves notice for its own sake, if only on account of the influential mathematicians who supported it.

The syndicate then proceeded to build a new scheme upon the ruins of the old. They considered that the result of the voting on the nineteen graces, in which the previous scheme had been submitted to the Senate, showed that it was the opinion of the University that the examination in Part III. should be independent of the preceding parts, and that no scheme would be acceptable in which it was not provided that all the subjects should be included in the examinations of each year. They accordingly presented a report to the Senate in October, 1878, in which they proposed that in June, immediately after the examination in Parts I. and II., the complete list of

wranglers, senior optimes, and junior optimes should be issued, arranged in order of merit; that Part III. should be a separate examination, to which wranglers only should be admissible; and that after the examination in Part III. a list should be issued in which the candidates were arranged in three classes, the names in each class being arranged in alphabetical order.

The schedule of subjects consisted of all the existing subjects divided into four groups, A., B., C., D. Group A. contained the pure mathematics; Group B., the astronomical subjects; Group C., hydrodynamics, sound, physical optics, elastic solids, &c.; and Group D., heat, electricity, and magnetism. In order to encourage the candidates to specialize their reading, one of the regulations authorised the Moderators and Examiners to place in the first division a candidate who showed eminent proficiency in any one group; so that it was not absolutely essential for a student, in order to be placed in the first division, to extend his reading beyond the subjects of a single group.

This scheme was approved by the Senate on November 21, 1878, and it came into operation in 1882, the first examination in Part III. taking place in 1883.

Parts I. and II. taken together differed in no essential respects from the Tripos as it had existed from 1848. The five days of the scheme of 1848 were reduced to three, and the range of subjects was more limited; otherwise the examination was exactly the same as in the period 1848–72. But Part III. was a complete novelty, and a great deal of curiosity was felt as to how the first Moderators and Examiners would interpret the regulations. Would the new examination resemble, as regards the character of the questions set, the last three days of the old five days, or was the examination to be one of a distinctly higher order? The result showed that the latter anticipation was the correct one. No longer hampered by the order of merit, the examiners felt themselves free to set difficult and elaborate questions, such as were only appropriate to specialists in the particular subjects; and a new departure was made.

As soon as the new system came into full operation, it was found that it needed amendment in various respects; and this is not to be wondered at, considering that it had been constructed in order to fit in with the few regulations that had escaped the general massacre of May 1878, and that almost every part of it was the result of a compromise. It was found that the interval between June and January —less than seven months, and including a long vacation in which very few lectures were given—was too short for an adequate preparation for Part III. It is true that most of the work for Part III.

could be done—and, indeed, was done—before the examination in Parts I. and II.; but the competition in these two parts remained as keen as ever, and, as the examination became imminent, the candidates were tempted to neglect the higher work, and give their whole attention to the more elementary subjects, upon which the list in order of merit depended. As a consequence, there was a diminution in the numbers of students attending the higher mathematical lectures in the University. With respect to the actual conduct of the examination, it was found that the strain upon the Moderators and Examiners was very serious, and general regret was expressed that under the new scheme no provision had been made for the annual appointment of an Additional Examiner, as in the previous scheme, which had been in operation from 1873 to 1882. Under the new system the candidates devoted themselves to special branches of the higher mathematics, and there was even greater difficulty in adequately representing all the subjects of examination. Accordingly, on June 12, 1884, the Senate confirmed a report of the Mathematical Board recommending that the examination in Part III. should take place in June, exactly a year after that in Parts I. and II., and that the Moderators and Examiners, with the Chairman of the Mathematical Board, should nominate an Additional Examiner, the first nomination being made in the Easter term, 1885, and having reference to the examination in January 1886. It was considered that the Moderators and Examiners were themselves the best judges of the branches of mathematics in which they most desired assistance, and were, therefore, the most suitable body to nominate the Additional Examiner.

The last time that the whole examination took place in January was in 1882. This year (1886) the examination in Part III. has taken place in January for the last time, so that the historic connection between the Tripos and the month of January has now finally ended. Henceforth the examination in all three parts will take place in the middle of the year.

When the interval between the earlier portion of the examination and Part III. had been extended to a year, it became evident that some substantial relief must be afforded to the examiners. By the existing regulations a person who accepted the office of Moderator would have to take part in the examination in three consecutive years, and in his second year of office he would have to examine the candidates of one year in Part III. simultaneously with those of the year below in Parts I. and II. This led to the consideration of the whole question of the appointment of examiners. The two Moderators in each year are nominated by two colleges, according to a prescribed cycle of fifty years. This nomination by colleges, though

theoretically not very defensible, had worked very fairly so long as the examination only included subjects with which any high wrangler might be expected to be acquainted; but it was clearly unsuitable for Part III. In any case, the nomination of the four examiners by four independent bodies might easily bring about the result that among the various subjects included there would be some which had not been made the object of special study by any of the examiners: indeed, there was nothing to prevent the four examiners being all pure mathematicians or all physicists. Accordingly, with a unanimity almost unique in matters relating to the Tripos, the Board recommended, in a report dated June 15, 1885, that the examiners for Part III. should be quite distinct from those for Parts I. and II., and that all four should be nominated by the Board. It was also proposed that they should hold office for only one year. This report was sanctioned by the Senate on October 29, 1886. In future, therefore, the Moderators will not take part in the highest portion of the examination. The appointment of Moderators dates from 1680. Previously the Proctors had themselves presided in the schools, but in that year the duty of conducting the disputations was transferred to the Moderators, who were specially appointed to perform this office. The Moderators have always been, and still remain, high University officers, ranking next to the Proctors.* Not only were they the earliest examiners in the University, but it is to them that we owe the origin of the examination system. Their severance from a portion—and that the highest portion—of the examination, is therefore a notable event in the history of the Tripos. Neither the Board nor the University would have agreed lightly to such a break in the traditions of the Senate House examination, had it been possible to retain the Moderators as examiners for the final part without altering the system of nomination by the colleges. The complete separation of Part III. from the earlier parts of the examination was, however, inevitable. Many members of the University who would discharge most admirably the duty of examining for Parts I. and II., would shrink from Part III.; and the professors and specialists who were best fitted to examine in Part III. would generally be reluctant to undertake the heavy burden of examining all the candidates for Parts I. and II., especially in two consecutive years.

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* It is still the custom to show the list of wranglers, senior optimes, and junior optimes, to the senior Proctor on the evening of the day before it is read in the Senate House. This custom doubtless had its origin in the fact that the Moderators were originally the substitutes of the Proctors. It has been already mentioned that the Proctors, as well as the Vice-Chancellor, used to have the right to insert a certain number of names where they pleased in the Tripos list.

Thus, by the irresistible pressure of events, it has come to pass, in the last few years, that not only the titles of wranglers, senior optimes, and junior optimes have lost their old significance, and refer only to the earlier examination, but that even the more ancient title and office of Moderator has undergone a similar restriction. The final part of the examination has indeed made rapid progress: within three years of its first coming into existence it has emancipated itself from union with the earlier parts, and become an independent examination.

Besides these important innovations, the Senate sanctioned at the same time a slight change in the nomenclature of the Tripos, the earlier portion of the examination, previously called Parts I. and II., upon which the list in order of merit depended, being designated Part I., and Part III. being henceforth designated Part II. This change was made in order to bring the nomenclature of the Mathematical Tripos into harmony with that of the Classical and other divided Triposes.

As soon as Part II., to adopt its new name, became an independent examination, the Board directed its attention to the schedule of subjects relating to it. The existing schedule contained only those subjects which had been included in the schedule which came into operation in 1873, when the results of the whole examination were still expressed by one final list, arranged in order of merit. Now that Part II. was a separate examination, and that there was no order of merit, the reasons for the limitation of the subjects had been entirely removed. Although the theory of elliptic functions, which dates only from the publication of Jacobi's "Fundamenta Nova" in 1829, was included, the theory of numbers, which had its origin in Gauss's "Disquisitiones Arithmeticæ" of 1801, was still excluded. Abelian functions, the theory of functions of a complex variable, projective geometry, and quaternions, were not formally included by name, and questions on these subjects could only be set, if at all, by straining the meaning of the title of some other subject. Besides the total exclusion of certain branches of pure mathematics, a further reason for revising the existing schedule was afforded by the fact that the four groups A., B., C., D. were very unequal, both in magnitude and popularity among the students. According to the existing regulations, the four groups had to be equally represented by questions, and this led to a great waste of examining power, many questions having to be constructed each year upon subjects which none of the candidates had studied. The Board accordingly formed a schedule in which all the subjects of pure and applied mathematics were included—none being intentionally omitted. These were divided into eight divisions, the first four relating to pure mathematics, and the last four to applied

mathematics. To avoid the waste of questions that would ensue from the examiners having to represent all the subjects in the papers of each year, they proposed that before the first day of December preceding the examination the names of the candidates and of the divisions and subjects in which they desired to be examined should be forwarded to the Registrary of the University. The examiners would thus be made acquainted with the subjects which the candidates had studied, and would be able to frame their questions accordingly. Changes were also proposed with respect to the candidates who were admissible, and to the form of the final list. By the existing regulations only wranglers were allowed to present themselves for examination in Part II., and the list was arranged in three divisions; there was no separation into *classes*, because, as only wranglers were admissible, it was considered that all the candidates were first-class men from the beginning. It was now proposed that the restriction which admits only wranglers should be removed, and that the candidates should be divided into three classes, each class being subdivided into as many divisions as the examiners in each year thought proper. In previous schemes the endeavour of the Board had been to frame regulations that would tempt the students to specialise their reading. A few years had made so great a difference that, with a view to prevent undue specialisation, the Board now inserted a regulation to the effect that proficiency in subjects taken from more than one of the divisions should be requisite in order that a candidate might be placed in the first class.

The report containing these proposals was confirmed by the Senate on May 27, and it is noteworthy that both this report and its predecessor, in which the nomination of examiners was placed in the hands of the Board, were sanctioned without opposition of any kind. The latter of these reports also made a few minor changes, the most important of which was the omission of the problem paper which had been still retained, from the old five days, in the scheme of 1882.* The examination in Part II. had assumed such a character that the kind of questions to which one would usually apply the name of problems was no longer in keeping with the contents of the other papers.†

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* Although the Board were unable to make any recommendation upon the subject, I may mention that the principle of prefixing to the final three days a preliminary day, in which the subjects of examination should be those parts of higher pure mathematics which are needed in mathematical physics, found a considerable amount of favour among the members of the Board. The proposal, however, was found to be more difficult in execution than was anticipated (partly on account of the impossibility of forming a perfectly satisfactory schedule of subjects for this day), and was ultimately abandoned by most of its original supporters.

† In the first examination in Part III., in 1883, the examiners set, as one of the question papers, a paper of essays; and their example was followed by the examiners in 1884, 1885, and 1886. These essay papers were introduced merely on the

Under the new scheme, in which all the examiners for Part II. were to be appointed by the Board with special reference to their collective fitness for conducting the examination, there was no further need for an Additional Examiner; this office was, accordingly, discontinued.*

Thus has the Mathematical Tripos been divided into two parts; and thus has surely arisen in the University a mathematical examination of a higher type than has been known before, or could have existed under any system in which all the candidates for mathematical honours were required to be examined by the same papers throughout. For those who study mathematics for the sake of exact knowledge or of mental discipline, and who propose to go forth into the world to follow professional or other careers, the first part secures all the old stimulus to industry, and gives to those who are successful the same stamp of intellectual distinction as before: such students are released at the end of their third year to enter upon the active duties of their lives, equipped with a sound understanding of the principles of the exact sciences, and with minds well trained to accurate habits in reasoning and in the acquisition of knowledge. To those whose attachment to our science lies deeper and whose studies have carried them beyond its threshold, the second part, at the end of their fourth year, affords an opportunity of distinction of a higher kind, and one more suited to their tastes: no longer is the wise and thoughtful student hopelessly distanced in the Tripos race by his quick and ready rival.

The wants of the candidate whose mathematical career closes with the last paper in Part I., and those of the candidate whose real mathematical life only begins from this moment, are equally provided for by the new scheme. The order of merit relates to an examination that can bear it. All the subjects included in Part I. are such as ought to be the common property of every one who has received a sound mathematical education; and by the results of an examination in subjects which all the candidates should have read, a list in order of merit can properly be formed. The specialist for the first time is set free to follow his own tastes, and to give his whole heart and time to the branches of mathematics by which he is attracted. The University permits him to select any subjects he pleases from the whole

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authority of the examiners, and not in consequence of any new regulation. Experience seems to show that the essay paper affords very little additional assistance in ascertaining the relative merits of the candidates. The essays were, perhaps, more useful at first, when they were a novelty.

* Unless the office should be revived at some future time, there will therefore have been only one Additional Examiner for the final part of the examination, viz. in 1886, the last occasion on which the examination took place in January.

range of pure and applied mathematics, and undertakes to examine him in them and award to him the credit he deserves for his attainments. A perfectly free choice is given to him, subject only to the one condition that, in order to qualify himself for admission to the first class, he must not select all his subjects from a single division.*

But what to us as mathematicians is more than all, as bearing on the future of our science, is that now for the first time will it be possible in Cambridge for an able and earnest worker and teacher to interest and engage his pupils in his work, and found a school such as we are so familiar with in foreign Universities, where the presence of a great professor has been almost invariably marked by a succession of illustrious pupils—pupils worthy of their master and worthy to carry on his work. Think of the school of arithmeticians founded by Gauss at Göttingen, and how impossible such a result would have been at Cambridge, dominated as she has been by the competition for places in the Tripos! Great as has been the value to the University of the order of merit—as a stimulus to industry, an encouragement to thoroughness in mathematical study, and a paramount influence in regulating elections to Fellowships at colleges where no independent examination existed—it has yet been in recent years a deadly enemy to the spread of research and the advance of our science. Throughout his whole career the student has had to devote himself unremittingly to the work for his Tripos, taking up a fresh subject each term, and often having to read two in one term. He could never pursue any subject far enough to reach the really interesting portions of it, or obtain complete command over its methods: he was always occupied with something new, starting afresh and gaining familiarity with new principles, new processes, new modes of thought. Many of the higher lectures in the University were necessarily neglected by the students: they could pay but scant attention to any subject which was not adequately represented in the Tripos, and even in the case of the subjects which were so represented they were tempted to pass lightly over those investigations, however important, which from their length and character were unsuitable for reproduction in an examination. Now, however, all this is history. When a good course of lectures upon any high subject is given in the University, those students who have attended the course will send in that subject as one in

* This condition would be complied with by the candidate's showing proficiency in one subject taken from one division, and in one other subject taken from another division. The intention of the Board was to discourage students from specialising too narrowly at too early a period. Some of the divisions (as, for example, the fourth, which contains only projective geometry and analytical geometry of curves and surfaces) are so restricted, that it was considered undesirable that students should be allowed to confine themselves entirely to subjects chosen from a single division.

which they desire to be examined: it will, therefore, be properly represented by questions; and the subject will become one that will be increasingly studied year by year. It will now be possible for any capable mathematician, by means of his lectures, to gather pupils round him who will bring his subject into prominence, and make it one of special study in the University.* It has been said that in mathematics we have in England generals without armies: the great men who are independent of circumstances have arisen among us, but where are the rank and file? It is my belief that the great obstacle to the existence of the rank and file has now been removed.

Whatever else it may be, Part II. is at all events a "limiting form." No wider choice of subjects could be given to the candidates; no greater freedom to the examiners. The schedule of subjects includes all mathematics: the examiners may issue any kind of list. By introducing numerous divisions into the classes, they may make it approximate as closely as they please to an order of merit; or, on the other hand, they may make it merely a class list. They are empowered to give to their list just such a form as they feel justified in doing by the results of the examination. In the appointment of examiners, also, the limiting form has been reached, all four being nominated by the same University authority, and holding office for one year only.

With respect to Part I., it is probable that the ultimate form has not yet been reached. There are some who think that, as in some other Triposes, the students should have the option of becoming candidates at the end of their second year. It would seem, also, that the range of subjects is too restricted; and, as may be inferred from what I have said near the beginning of my address, I should myself like to see the elementary portions of elliptic functions included in the schedule of Part I. Other alterations in the schedules might also be suggested. Still, these are but minor points; and I think that the main principle of subjecting all the candidates for mathematical honours to one and the same examination in comparatively elementary subjects, and arranging the list in order of merit, meets with general approval.

A few years ago, when the old Tripos was exerting its stifling influence upon the higher mathematical studies of the Universities, I felt disposed to welcome the abolition of the order of merit as the lesser of two evils; but now that the Tripos is divided, and that the mathematician has his own examination especially framed for him, I should be sorry to see a modified class list substituted for the order

* In the schedule for Part II. no subjects are ignored or favoured less than others, so that by the new scheme provision is made for the growth of any subject which may happen to take root.

of merit in Part I. A severe competition for places has the great advantages of keeping the candidates closely employed, and extracting from them their best work. At present an immense amount of thoroughly good mathematical work is done in the University. We have received from our predecessors a system under which the principles of mathematics are efficiently taught, the powers of the students are exerted to the utmost, and upwards of a hundred persons each year receive a mathematical education which is in some respects unique. These are substantial advantages, which should not lightly be jeopardised or exchanged for others that are problematical. Under any other system I think the quantity and the quality of the mathematical work done in the University would suffer. It should be remembered also that there is no subject in which the knowledge of a candidate can be so readily tested by examination as in mathematics, and that in no other subject can the results of an examination be expressed with such certainty and accuracy by an order of merit.

I believe there are, indeed, but very few who have graduated in the Tripos who would set a slight value upon the advantage which their mathematical training has been to them throughout life ; and, on the other hand, I think that it has been an indirect benefit to our science that among those who have won distinction in public and professional life there have always been some—and those not the least influential or eminent—who have passed through an extensive and thorough course of mathematical study, and to whom our world of symbols is no *terra incognita*. The fact that our results, unlike the conquests of astronomy and other branches of applied mathematics, can only be expressed by means of a language of their own, requiring years of study, imposes of necessity such narrow limitations upon the numbers of our audience that we cannot be insensible to the advantages of any system by which the power of understanding and appreciating the beauties of our science is extended to others external to our own ranks. Under the new scheme these advantages are still retained ; and, difficult as is the problem of combining a mathematical course for the many with the technical requirements of the few, I believe that a satisfactory solution has rewarded the efforts of the last twenty years. I believe that the University of Cambridge will become a great centre of mathematical research and a home of the exact sciences, and that it will be found that these objects have been attained without any sacrifice of the general efficiency of the training received by the bulk of the candidates for mathematical honours.

On taking a survey of the history of the Tripos during the last half century, perhaps the feature that stands out most strongly is the part played by the subjects of electricity and magnetism—their half-

recognised existence before 1848, their subsequent exclusion until 1873, and the effects which followed their restoration in that year. It was the extension of the dominion of mathematics over these great and growing branches of physical science that broke down the old system. Electricity and magnetism became too important to be excluded; but when included, the examination in its old form was too heavily weighted to exist.

The year 1877–78, in which the syndicate of 1877 was endeavouring to frame a scheme that should relieve the strain of the excessive competition without sacrificing the order of merit, was perhaps the most eventful period in the whole history of the examination: it then became evident that it was impossible to retain the existing system, even in a modified form, and that a complete reorganisation of some kind was inevitable. Although the frequent changes in the last few years have been productive of some inconvenience, I think it is fortunate that the syndicate was so reluctant to propose any sweeping changes, and that the present scheme has come into existence as it has done—not as the work of any influential legislator, but as the form which the examination has of itself assumed under the pressure of the actual forces at work in the University. The order of merit for the whole examination was not given up till it was clearly shown that its retention was an impossibility; and, on the other hand, Part II. has grown up by a process of regular development, and has been moulded into its present form by those most interested in promoting the higher mathematical studies of the University.

Special reference also should be made to the "three days." It will be remembered that this preliminary portion of the examination was the principal feature of the scheme which came into operation in 1848. Both the subjects and the methods of solution that may be employed are defined by a schedule, and only those who satisfy the examiners by their performances in these three days are admitted to the subsequent parts of the examination. It is very singular that an arrangement devised so long ago should not only still continue in force, but even be regarded by some as the most thoroughly satisfactory portion of the whole system. The framers of this scheme and schedule might well have been proud of the lasting character of their work if they could have known that it would outlive two sets of University statutes, and, amidst changes on every side, remain unchanged for forty years.

The early history of the Tripos, and its gradual development into an examination by written answers, and finally by printed papers also, are especially interesting in these days, when the merits of the examination system are so highly appreciated, and its adoption

is so universal. The Senate House at Cambridge is the cradle of the modern form of examination in England.

In connection with the Tripos there is one matter of so much importance that I cannot pass it over entirely without mention. I mean the influence of the system of private tuition. I believe that while there is an order of merit it will always be a great assistance to the majority of the candidates to read with a private tutor. Mathematics is a difficult science; and, when a considerable range of subjects has to be traversed in a comparatively short time, and the knowledge of the candidate has to be finally tested by an examination such as Part I. of the Tripos, it cannot fail to be of great advantage to him to have his difficulties explained, his path smoothed, and his skill in working out problems developed, by an experienced private tutor specially interested in his individual welfare. The system of private tuition has been objected to from two points of view: (1) because it is unsatisfactory that the instruction which is valued most highly by the student should be received from his private tutor instead of from college lecturers or University professors; and (2) because the student who has followed implicitly, during his whole undergraduate career, the minute directions of one man with regard to his reading, is placed after his degree, when he is deprived of his guide, in a very unfavourable position for pursuing further his mathematical studies. The first objection does not concern us here: my own feeling is, as I have just said, that, whenever an order of merit exists and the competition for places is keen, the services of private tutors will necessarily be called into requisition. The second objection is one which is of far more importance to our science. There can be no question that, brilliant and eminent as have been the greatest of the private tutors at Cambridge, one result of the system has been that many of the ablest students have been left after graduation, not only without any knowledge of the way to follow up the study of the subjects of which they had learned the elements, but even without any taste or inclination to do so. The private tutor's manuscript and verbal instruction had superseded all need of referring to the original memoirs, and the nascent wrangler knew nothing of the great world of mathematical literature or of the modes of reaching it. On the other hand, it is only fair to say that the amount of mathematical knowledge acquired by the best pupils from their private tutors in the course of their undergraduate career was really wonderful; and that, till quite recently, neither the University nor the colleges offered any inducement to the mathematical student to continue his reading after the Tripos. The fact that the student's horizon was bounded by the Tripos, and that his training was directed with a view to giving him

skill in working out questions rather than to developing his taste for the science he was studying, was principally the fault of the system as a whole; but it was certainly intensified by the complete subjection of the pupil to the course of reading placed before him by the private tutor. A student whose interests and aspirations had been at least held in check, and perhaps entirely stifled, throughout his whole undergraduate career, was generally too subdued or helpless to be able to make use of his freedom when the examination was over.

Under the new scheme the private tutor still occupies in the main his old position with respect to Part I., although, of course, the higher places in this examination have much less significance than before. With respect to Part II., it is quite different. No attempt is made by private tutors to teach these higher subjects, which, both from their character and extent, are clearly unsuited for private tuition; and the students are compelled to rely upon the lectures delivered in the colleges and University in their preparation for this final examination. Thus, in their fourth year, they are brought into contact with the leading mathematicians in Cambridge; and when the examination sets them free to pursue their own studies or researches, they start on their new career fresh from the best and highest teaching which the University affords.

Although the subject of my address is the Mathematical Tripos, it may be regarded as still falling within my province to refer to other changes that have taken place in the University for the purpose of encouraging original mathematical work. Fourteen years ago Trinity College invited mathematical candidates for Fellowships to send in, before the examination, dissertations upon any subjects of their own selection. It was announced that these dissertations, if possessed of decided merit, would be taken into account in the Fellowship election, together with the results of the Fellowship examination. Not only have these dissertations been of the greatest value in guiding the choice of the electors, but many of them have been important contributions to mathematical literature.* The example of Trinity College has been recently followed by St. John's College and by King's College. The Smith's Prizes, which for a great number of years had been awarded by a special examination, are now awarded

* Among the Trinity dissertations which have subsequently been printed, I may mention the late Mr. R. C. Rowe's "Memoir on Abel's Theorem" (*Phil. Trans.*, 1881), Mr. Forsyth's "Memoir on the Theta Functions" (*Phil. Trans.*, 1882), Mr Homersham Cox's "Application of Quaternions and Grassmann's Ausdehnungslehre to different kinds of Uniform Space" (*Camb. Trans.*, 1882), Mr. Gallop's "Distribution of Electricity on the Circular Disc and Spherical Bowl" (*Quart. Math. Journ.*, 1886), and Mr. R. Lachlan's "Systems of Circles and Spheres" (*Phil. Trans.*, 1886).

annually for mathematical dissertations, the candidates being free to select their own subjects. This new scheme passed the Senate on October 25, 1883, and the first award of the prizes under it was made in 1885. Powerful inducements are, therefore, now held out by the University and by some of the colleges for the best students to devote themselves to original work. The importance to our science of these direct incentives to research cannot be over-estimated. They come into operation as soon as the stimulus of the examination is removed, and, instead of resting upon their laurels, the ablest mathematicians of the year are induced to concentrate their powers upon a single subject, just at the time when they are undaunted by any amount of hard work, when their stock of general mathematical knowledge is freshly acquired, and when their minds are flexible, vigorous, and free from care. It is, indeed, strange to look back upon the changes of the last few years, and to contrast the encouragement now given to mathematical research with the indifference, or even worse, of twenty years ago.*

I cannot close my address without saying a few words upon our Society. We were founded in 1865, and so we have now attained our majority. I think we can safely say that we have steadily and uniformly kept to our single purpose of promoting the advance of mathematics. We have published seventeen volumes of *Proceedings*, and every paper we have printed has been subjected to a rigorous examination by two referees. We have already a history to look back upon: familiar presences among us—De Morgan, Clerk Maxwell, Clifford, Henry J. S. Smith, Spottiswoode—have passed away; and for most of us this very room is full of associations with those whom we shall see no more. I should like, before concluding, formally to express our gratitude to our two Secretaries, Mr. Morgan Jenkins and Mr. Robert Tucker, who have served us so faithfully almost from our foundation, and to whom the successful development of our Society has been largely due. I will not utter any aspirations with regard to our future: we shall never be a great Society in numbers, but we can continue to do what we have done, and to spare no effort to encourage the advance of mathematical science.

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* Under the old system, the Cambridge graduate who devoted himself to mathematical research possessed one advantage over his Continental colleagues, in the wider range of his general mathematical knowledge. Although Part I. is considerably more restricted than the Tripos of 1848—72, this advantage is still retained to a substantial extent.

*On Certain Operators in connection with Symmetric Functions.**

By R. Lachlan, B.A.

[*Read November 11th*, 1886.]

The present paper arose out of an attempt to deduce, by a symbolical method, the generalised formula, given by Captain Macmahon (*Proceedings*, Vol. xv., p. 39), as an extension of Newton's formula for the sums of the powers of the roots of a given equation. It afterwards occurred to the author that several other formulæ connected with symmetric functions might be very easily deduced by the same method.

1. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots + (-)^n p_n = 0;$$

let s_r denote, as usual, the sum of the r^{th} powers of the roots, and let h_r denote the total symmetric function whose weight is r, so that

$$\begin{aligned}(1-\alpha_1 x)(1-\alpha_2 x) \ldots (1-\alpha_n x) &= 1 - p_1 x + p_2 x^2 - \ldots + (-)^n p_n x^n \\ &= \frac{1}{1 + h_1 x + h_2 x^2 + \ldots}.\end{aligned}$$

The operator most employed is

$$\alpha_1^2 \frac{\partial}{\partial \alpha_1} + \alpha_2^2 \frac{\partial}{\partial \alpha_2} + \ldots = -\Sigma \frac{\partial}{\partial \frac{1}{\alpha}},$$

which is denoted by σ, since usually the function operated on will be supposed expressed in terms of s_1, s_2, &c.

2. We have at once, from the definition,

$$\sigma s_r = \Sigma \alpha^2 \frac{\partial}{\partial \alpha} \alpha^r = r s_{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

To find σp_r, let

$$(1-\alpha_2 x)(1-\alpha_3 x) \ldots (1-\alpha_n x) = 1 - p_1' x + p_2' x^2 - \ldots;$$

* At the suggestion of one of the Referees, this paper has been re-written for the purpose of demonstrating some of the formulæ more fully.

so that $1-p_1x+p_2x^2-\ldots = (1-a_1x)(1-p_1'x+p_2'x^2-\ldots)$,

then we have $$p_r = p_r'+a_1p_{r-1}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

hence $$\frac{\partial}{\partial a_1}p_r = p_{r-1}',$$

and so we may write (2),

$$p_r = \frac{\partial}{\partial a_1}p_{r+1}+a_1\frac{\partial}{\partial a_1}p_r;$$

hence $$p_r\Sigma a = \Sigma a\frac{\partial}{\partial a}p_{r+1}+\Sigma a^2\frac{\partial}{\partial a}p_r;$$

therefore $$\sigma p_r = s_1p_r-(r+1)\,p_{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

To find σh_r, let

$$\frac{1}{(1-a_2x)\ldots(1-a_nx)} = 1+h_1'x+h_2'x^2+\ldots$$
$$= (1-a_1x)(1+h_1x+h_2x^2+\ldots);$$

then we have $$h_{r+1}-a_1h_r = h_{r+1}',$$

and $$\frac{\partial}{\partial a_1}h_{r+1} = h_r+a_1\frac{\partial}{\partial a_1}h_r;$$

thus, $$\Sigma a\frac{\partial}{\partial a}h_{r+1} = h_r\Sigma a+\Sigma a^2\frac{\partial}{\partial a}h_r,$$

whence $$\sigma h_r = (r+1)\,h_{r+1}-s_1h_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

3. Now let π, ϑ be two operators defined by means of the equations

$$\sigma = s_1-\pi = \vartheta-s_1,$$

then it follows, from (3) and (4), that

$$\pi p_r = (r+1)\,p_{r+1},$$
$$\vartheta h_r = (r+1)\,h_{r+1};$$

also, we have $$2\sigma = \vartheta-\pi.$$

It may be noticed that, if c be any constant,

$$\pi^r c = r!\,cp_r, \quad \vartheta^r c = r!\,ch_r,$$

but $$\pi^r cu = c\pi^r u, \quad \vartheta^r cu = c\vartheta^r u.$$

4. From the definition of σ, we have at once

$$\sigma(uv) = v\sigma u+u\sigma v,$$

and, generally, $$\sigma^m (uv) = (\sigma_1+\sigma_2)^m uv \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5);$$

where σ_1, σ_2 operate only on u, v, respectively. Similarly,

$$\sigma^m (u_1 u_2 \ldots u_\kappa) = (\sigma_1+\sigma_2+\ldots+\sigma_\kappa)^m u_1 u_2 \ldots u_\kappa,$$

where σ_κ operates only on u_κ. Thus we have

$$\sigma^m (a_1 a_2 \ldots a_\kappa) = \Sigma \frac{m!}{\alpha!\,\beta!\ldots\mu!}\, \sigma_1^\alpha \sigma_2^\beta \ldots \sigma_\kappa^\mu (a_1 a_2 \ldots a_\kappa)$$

$$= m!\, \Sigma a_1^{1+\alpha} a_2^{1+\beta} \ldots a_\kappa^{1+\mu},$$

where $\alpha+\beta+\ldots+\mu = m$; hence it follows that

$$\sigma^m p_\kappa = m!\, S^\kappa_{m+\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where $S^\kappa_{m+\kappa}$ is the sum of the symmetric functions of weight $m+\kappa$, each containing κ, and only κ, roots.

5. We have, from the definition of π,

$$\pi u = s_1 u - \sigma u,$$

therefore
$$\pi^2 u = s_1^2 u - s_1 \sigma u + \sigma^2 u - \sigma (s_1 u)$$
$$= s_1^2 u - 2 s_1 \sigma u + \sigma^2 u - u \sigma s_1$$
$$= 2p_2 u - 2p_1 \sigma u + \sigma^2 u.$$

Let us assume that $$\pi^r u = (p-\sigma)^r u,$$

where, after expanding, p^κ is to be replaced by $\kappa!\, p_\kappa$; then we have

$$\pi^{r+1} u = (s_1-\sigma)\, \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!}\, p_{r-\kappa} \sigma^\kappa u$$

$$= \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} (s_1 p_{r-\kappa} - \sigma p_{r-\kappa})\, \sigma^\kappa u$$

$$- \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!}\, p_{r-\kappa} \sigma^{\kappa+1} u$$

$$= \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} (r+1-\kappa)\, p_{r+1-\kappa} \sigma^\kappa u$$

$$+ \Sigma_\kappa (-)^{\kappa+1} \frac{r!}{\kappa!}\, p_{r-\kappa} \sigma^{\kappa+1} u$$

$$= \Sigma_\kappa (-)^\kappa \frac{(r+1)!}{\kappa!}\, p_{r+1-\kappa} \sigma^\kappa u$$

$$= (p-\sigma)^{r+1} u.$$

Hence, for all integral values of r,

$$\pi^r u = (p-\sigma)^r u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

where, after expanding, p^κ is to be replaced by $\kappa!\, p_\kappa$.

More generally, if $f(z)$ be a rational integral function of z, we shall have

$$f(\pi)\, u = f(p-\sigma)\, u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

where p^κ is to be replaced by $\kappa!\, p_\kappa$, after expanding.

6. From (7), we have $\pi^r p_1 = (p-\sigma)^r s_1$,

hence
$$(r+1)!\, p_{r+1} = \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} p_{r-\kappa} \sigma^\kappa s_1$$
$$= r!\, \Sigma_\kappa (-)^\kappa p_{r-\kappa} s_{\kappa+1},$$

or
$$(r+1)\, p_{r+1} = p_r s_1 - p_{r-1} s_2 + \ldots + (-)^r s_{r+1} \ldots\ldots\ldots\ldots (9);$$

which is Newton's formula.

Again, from (7), we have

$$\pi^r p_m = (p-\sigma)^r p_m,$$

hence
$$\frac{(r+m)!}{m!} p_{r+m} = \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} p_{r-\kappa} \sigma^\kappa p_m$$
$$= r!\, \Sigma_\kappa (-)^\kappa p_{r-\kappa} S^m_{m+\kappa};$$

thus we have

$$\frac{(r+m)!}{r!\, m!} p_{r+m} = p_r S^m_m - p_{r-1} S^m_{m+1} + \ldots + (-)^r S^m_{m+r} \ldots\ldots\ldots\ldots (10);$$

which is Captain Macmahon's extension of (9) [*Proceedings*, Vol. xv., p. 39, (57)].

7. Again, we have, from the definition of ϑ,

$$\vartheta u = s_1 u + \sigma u,$$

and, by an exactly similar proof to that in § 5, we can deduce that

$$\vartheta^r u = (h+\sigma)^r u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where, after expanding, h^κ is to be replaced by $\kappa!\, h_\kappa$.

8. From (11), we have

$$\vartheta^r h_1 = \Sigma_\kappa \frac{r!}{\kappa!} h_{r-\kappa} \sigma^\kappa s_1,$$

therefore $$(r+1)\,h_{r+1} = h_r s_1 + h_{r-1} s_2 + \ldots + s_{r+1} \quad \ldots\ldots\ldots\ldots\ldots (12).$$

9. Since $$\sigma\,(uv) = u\sigma v + v\sigma u,$$

we have

$$\pi\,(uv) - u\pi v - v\pi u = s_1 uv - \sigma\,(uv) - us_1 v + u\sigma v - vs_1 u + v\sigma u$$
$$= -\,s_1 uv,$$

or $$\pi\,(uv) = u\pi v + v\pi u - s_1 uv \quad \ldots\ldots\ldots\ldots\ldots\ldots (13);$$

and, similarly,

$$\pi\,(u_1 u_2 \ldots u_m) = \{\pi_1 + \pi_2 + \ldots + \pi_m - (m-1)\,s_1\}\,(u_1 u_2 \ldots u_m) \ldots (14),$$

where π_κ operates only on u_κ.

Thus we have $$\pi u^m = mu^{m-1}\pi u - (m-1)\,s_1 u^m.$$

10. Again, from (13),

$$\pi^2\,(uv) = u\pi^2 v + 2\pi u\,.\,\pi v + v\pi^2 u - 2s_1\,(u\pi v + v\pi u) - uv\,\pi s_1 + uv\,.\,2s_1^2;$$

but $$2s_1^2 - \pi s_1 = 2p_1^2 - 2p_2 = 2h_2;$$

hence $$\pi^2\,(uv) = (\pi_1 + \pi_2 - h)^2\,uv,$$

where, after expanding, h^κ is to be replaced by $\kappa!\,h_\kappa$; and π_1, π_2 only operate on u, v, respectively.

Let us assume that

$$\pi^r\,(uv) = (\pi_1 + \pi_2 - h)^r\,uv$$
$$= \Sigma\,(-)^\kappa \frac{r!}{\lambda!\,\mu!}\,h_\kappa\,(\pi^\lambda u)(\pi^\mu v),$$

where $\lambda + \mu + \kappa = r$.

Then, by (14),

$$\pi^{r+1}\,(uv) = \Sigma\,(-)^\kappa \frac{r!}{\lambda!\,\mu!}\,(\pi - 2s_1)\,h_\kappa\,(\pi^\lambda u)(\pi^\mu v)$$
$$+ \Sigma\,(-)^\kappa \frac{r!}{\lambda!\,\mu!}\,h_\kappa\,\{(\pi^{\lambda+1} u)(\pi^\mu v) + (\pi^\lambda u)(\pi^{\mu+1} v)\};$$

and, by our definition, $$\vartheta - s_1 = s_1 - \pi,$$

so that $$(\pi - 2s_1)\,h_\kappa = -\,\vartheta h_\kappa = -\,(\kappa+1)\,h_{\kappa+1};$$

hence $$\pi^{r+1}(uv) = \Sigma(-)^{\kappa+1}\frac{r!\,(\kappa+1)}{\lambda!\,\mu!}h_{\kappa+1}(\pi^{\lambda}u)(\pi^{\mu}v)$$

$$+\Sigma(-)^{\kappa}\frac{r!}{\lambda!\,\mu!}h_{\kappa}(\pi^{\lambda+1}u)(\pi^{\mu}v)$$

$$+\Sigma(-)^{\kappa}\frac{r!}{\lambda!\,\mu!}h_{\kappa}(\pi^{\lambda}u)(\pi^{\mu+1}v).$$

In this the coefficient of the term,

$$h_{\kappa+1}(\pi^{\lambda}u)(\pi^{\mu}v),$$

$$= (-)^{\kappa+1}\frac{r!}{\lambda!\,\mu!}\{(\kappa+1)+\lambda+\mu\} = (-)^{\kappa+1}\frac{(r+1)!}{\lambda!\,\mu!};$$

consequently $$\pi^{r+1}(uv) = (\pi_1+\pi_2-h)^{r+1}(uv);$$

and thus, whatever integral value r may have, the theorem holds, viz.,

$$\pi^{r}(uv) = (\pi_1+\pi_2-h)^{r}(uv) \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

where π_1 operates only on u, π_2 only on v, and where h^{κ} is to be replaced by $\kappa!\,h_{\kappa}$, after expanding.

11. Again, we have

$$\vartheta(uv) = u\vartheta v + v\vartheta u - s_1 uv,$$

and, exactly as in (13),

$$\vartheta(u_1u_2\ldots u_m) = \{\vartheta_1+\vartheta_2+\ldots+\vartheta_m-(m-1)s_1\}(u_1u_2\ldots u_m)\ldots(16),$$

where ϑ_{κ} operates only on u_{κ}.

Again, $$\vartheta^2(uv) = u\vartheta^2 v + 2\vartheta u\,.\,\vartheta v + v\vartheta^2 u - 2s_1(u\vartheta v + v\vartheta u) - uv\vartheta s_1 + uv\,2s_1^2,$$

or, since $$2s_1^2 - \vartheta s_1 = 2h_1^2 - 2h_2 = 2p_2,$$

we have $$\vartheta^2(uv) = (\vartheta_1+\vartheta_2-p)^2\,uv;$$

and, proceeding exactly as in § 10, we can prove that

$$\vartheta^{r}(uv) = (\vartheta_1+\vartheta_2-p)^{r}\,uv \quad \ldots\ldots\ldots\ldots\ldots\ldots(17),$$

where ϑ_1 operates only on u, ϑ_2 only on v, and where p^{κ} is to be replaced by $\kappa!\,p_{\kappa}$ after expanding.

12. Again, since $$\sigma u = (s_1-\pi)\,u,$$

therefore $$\sigma^2 u = s_1^2 u - 2s_1\pi u + \pi^2 u - u\pi s_1 + s_1^2 u,$$

and $$2s_1^2 - \pi s_1 = 2h_2,$$

therefore $$\sigma^2 u = (h-\pi)^2 u,$$

where h^κ is to be replaced by $\kappa!\, h_\kappa$, after expanding.

Assume then, that $$\sigma^r u = (h-\pi)^r u$$

$$= \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} h_{r-\kappa} \pi^\kappa u;$$

then we have $$\sigma^{r+1} u = \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} \sigma h_{r-\kappa} \pi^\kappa u$$

$$+ \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} h_{r-\kappa} \sigma (\pi^\kappa u).$$

But $$\sigma h_{r-\kappa} = \vartheta h_{r-\kappa} - s_1 h_{r-\kappa},$$

$$\sigma (\pi^\kappa u) = s_1 \pi^\kappa u - \pi^{\kappa+1} u;$$

hence $$\sigma^{r+1} u = \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} (r+1-\kappa) h_{r+1-\kappa} \pi^\kappa u$$

$$- \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} h_{r-\kappa} \pi^{\kappa+1} u$$

$$= \Sigma_\kappa (-)^\kappa \frac{(r+1)!}{\kappa!} h_{r+1-\kappa} \pi^\kappa u$$

$$= (h-\pi)^{r+1} u.$$

Hence, for all integral values of r,

$$\sigma^r u = (h-\pi)^r u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

where, after expanding, h^κ is to be replaced by $\kappa!\, p_\kappa$.

13. Similarly, we shall have, since

$$\sigma u = \vartheta u - s_1 u,$$

$$\sigma^r u = (\vartheta - p)^r u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

where p^κ is to be replaced by $\kappa!\, p_\kappa$, after expanding.

14. From (18), we have

$$\sigma^r s_1 = (h-\pi)^r p_1,$$

hence $$r!\, s_{r+1} = \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} h_{r-\kappa} \pi^\kappa p_1$$

$$= \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!} h_{r-\kappa} (\kappa+1)!\, p_{\kappa+1},$$

therefore $s_{r+1} = p_1 h_r - 2p_2 h_{r-1} + 3p_3 h_{r-2} - \ldots + (-)^r (r+1)\, p_{r+1} \ldots (20)$;

and similarly, from (19),

$$s_{r+1} = (r+1)\, h_{r+1} - r p_1 h_r + (r-1)\, p_2 h_{r-1} - \ldots + (-)^r h_1 p_r \ldots (21).$$

15. Again, from (18), $\sigma^r p_m = (h-\pi)^r p_m$,

therefore $$r!\, S^m_{m+r} = \Sigma_\kappa (-)^\kappa \frac{r!}{\kappa!}\, h_{r-\kappa} \pi^\kappa p_m,$$

therefore $$S^m_{m+r} = \Sigma_\kappa (-)^\kappa \frac{(\kappa+m)!}{\kappa!\; m!}\, h_{r-\kappa}\, p_{m+\kappa} \quad \ldots\ldots\ldots\ldots (22);$$

a formula marked (56) in Capt. Macmahon's paper (*Proceedings*, Vol. xv.).

16. From (5), we have

$$\sigma^m s_1^2 = \Sigma \frac{m!}{(m-\kappa)!\;\kappa!}\, \sigma^{m-\kappa} s_1 \,.\, \sigma^\kappa s_1$$

$$= m!\, \Sigma\, s_{m+1-\kappa}\, s_{\kappa+1};$$

but $$\sigma^m s_1^2 = \sigma^m s_2 + 2\sigma^m p_2$$

$$= (m+1)!\; s_{m+2} + 2m!\; S^2_{m+2};$$

hence $$2S^2_{m+2} + (m+1)\, s_{m+2} = s_1 s_{m+1} + s_2 s_m + \ldots + s_{m+1} s_1 \ldots\ldots\ldots (23).$$

Again, since $$s_1 p_r = \sigma p_r + (r+1)\, p_{r+1},$$

$$\sigma^m s_1 p_r = \sigma^{m+1} p_r + (r+1)\, \sigma^m p_{r+1}$$

$$= (m+1)!\; S^r_{m+1+r} + (r+1)\, m!\; S^{r+1}_{m+r+1}.$$

But, by (5), $$\sigma^m s_1 p_r = \Sigma \frac{m!}{(m-\kappa)!\;\kappa!}\, \sigma^{m-\kappa} s_1 \,.\, \sigma^\kappa p_r$$

$$= \Sigma \frac{m!}{(m-\kappa)!\;\kappa!}\, (m-\kappa)!\; s_{m+1-\kappa} \,.\, \kappa!\; S^r_{\kappa+r};$$

hence $$(r+1)\, S^{r+1}_{m+r+1} + (m+1)\, S^r_{m+r+1}$$

$$= s_1 S^r_{m+r} + s_2 S^r_{m+r-1} + s_3 S^r_{m+r-2} + \ldots + s_{m+1} S^r_r \ldots\ldots (24).$$

17. Several curious identities may be deduced from the formulæ (15) and (17). Thus, since

$$p_1^2 = s_2 + 2p_2,$$

we have, by (15),

$$\pi^r(s_2+2p_2) = \Sigma(-)^\gamma \frac{r!}{\alpha!\,\beta!} h_\gamma(\pi^\alpha p_1)(\pi^\beta p_1)$$

$$= r!\,\Sigma(-)^\gamma(\alpha+1)(\beta+1)\,p_{\alpha+1}p_{\beta+1}h_\gamma,$$

where $$r = \alpha+\beta+\gamma.$$

But $$\pi^r s_2 = (p-\sigma)^r s_2$$

$$= \Sigma(-)^\kappa \frac{r!}{\kappa!}\, p_{r-\kappa}\sigma^\kappa s_2$$

$$= \Sigma(-)^\kappa r!\, p_{r-\kappa}(\kappa+1)\, s_{\kappa+2},$$

and $$\pi^r(2p_2) = 2\frac{(r+2)!}{2}p_{r+2}.$$

Hence $$(r+1)(r+2)\,p_{r+2}+\Sigma(-)^\kappa(\kappa+1)\,p_{r-\kappa}s_{\kappa+2}$$

$$= \Sigma(-)^\gamma(\alpha+1)(\beta+1)\,p_{\alpha+1}p_{\beta+1}h_\gamma \ldots\ldots\ldots (25);$$

where $\alpha+\beta+\gamma = r$, and κ, α, β, γ have all values from 0 to r inclusive.

A similar formula may be obtained from (17).

18. It would be easy to express p_r, h_r, s_r in terms of p_1, p_2, &c., h_1, h_2, &c., or s_1, s_2, &c., by means of these operators.

Thus, to express p_r in terms of s_1, s_2, &c., we have

$$r!\,p_r = \pi(r-1)!\,p_{r-1}$$

$$= (s_1-\sigma)(r-1)!\,p_{r-1};$$

hence we have, in succession,

$$p_1 = s_1,$$

$$2\,p_2 = s_1^2-\sigma s_1 = s_1^2-s_2,$$

$$3!\,p_3 = s_1^3-s_1s_2-2s_1s_2+2s_3$$

$$= s_1^3-3s_1s_2+2s_3,$$

$$4!\,p_4 = s_1^4-3s_1^2s_2+2s_1s_3$$

$$-3s_1^2s_2+3s_2^2+6s_1s_3-6s_4$$

$$= s_1^4-6s_1^2s_2+8s_1s_3+3s_2^2-6s_4,$$

... ...

Or, again, to express h_r in terms of p_1, p_2, &c., we have

$$r!\,h_r = \vartheta(r-1)!\,h_{r-1}$$

$$= (2p_1-\pi)(r-1)!\,h_{r-1};$$

hence we have $h_1 = p_1$;

therefore $2!\, h_2 = 2p_1^2 - 2p_2$,

or $h_2 = p_1^2 - p_2$;

therefore $3h_3 = 2p_1^3 - 2p_1p_2 - 4p_1p_2 + p_1^3 + 3p_3$,

or $h_3 = p_1^3 - 2p_1p_2 + p_3$;

therefore $4h_4 = 2p_1^4 - 4p_1^2p_2 + 2p_1p_3 - 6p_1^2p_2 + 2p_1^4$
$+ 4p_2^2 + 6p_1p_3 - 2p_1^2p_2 - 4p_4$,

or $h_4 = p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4$,

... ...

On the Transformations of the General Elliptic Element

$$\frac{\delta_x}{\sqrt{U_x}},$$

where $U_x = x-\alpha\,.\,x-\beta\,.\,x-\gamma\,.\,x-\delta = ax^4 + 4bx^3 + 6cx^2 + 4dx + e.$

By Mr. R. Russell, M.A.

[*Read Nov. 11th*, 1886.]

I. *Transformation of First Order.*—If in the element we replace x by $\frac{ly+m}{l'y+m'}$, we get one of similar type, in which the quartic inside the radical is the result of transforming U_x by the substitution $x = \frac{ly+m}{l'y+m'}$. Hence, if θ_1, θ_2 be roots of one of the harmonic factors of sextic covariant, then, if $y = p\,\frac{x-\theta_1}{x-\theta_2}$, we must have

$$\frac{\delta_x}{\sqrt{U_x}} = \frac{k\,dy}{\sqrt{Ay^4 + 6Cy^2 + E}}.$$

There are thus, in all, six ways of reducing the element by a linear

substitution to its canonical form, viz.,

$$py = \frac{x-\theta_1}{x-\theta_2}, \quad \frac{x-\theta_2}{x-\theta_1}, \quad \frac{x-\theta_2}{x-\theta_3}, \quad \frac{x-\theta_3}{x-\theta_2}, \quad \frac{x-\theta_3}{x-\theta_1}, \quad \frac{x-\theta_1}{x-\theta_3}.$$

To find that linear transformation which leaves the quartic unchanged. Obviously, when $x = \alpha, \beta, \gamma, \delta$; $y = \beta, \alpha, \delta, \gamma$; therefore, if $y = -\frac{lx+m}{l'x+m'}$, we have

$$l'\alpha\beta + l\alpha + m'\beta + m = 0,$$
$$l'\alpha\beta + l\beta + m'\alpha + m = 0,$$
$$l'\gamma\delta + l\gamma + m'\delta + m = 0,$$
$$l'\delta\gamma + l\delta + m'\gamma + m = 0,$$

therefore $$l = m',$$

therefore $$l'\alpha\beta + l(\alpha+\beta) + m = 0,$$
$$l'\gamma\delta + l(\gamma+\delta) + m = 0,$$

therefore $$\frac{l'}{\alpha+\beta-\gamma-\delta} = \frac{l \text{ or } m'}{\gamma\delta-\alpha\beta} = \frac{m}{\alpha\beta(\gamma+\delta)-\gamma\delta(\alpha+\beta)}.$$

Hence $y = -\frac{Z_2}{Z_1}$, where Z_1 and Z_2 are the differentials of one of the harmonic factors above mentioned.

There are, therefore, three different transformations of this kind.

I shall now make a short digression on the factors of the sextic covariant. X, Y, Z are defined as follows :—

$$\frac{1}{x-\beta} + \frac{1}{x-\gamma} - \frac{1}{x-\alpha} - \frac{1}{x-\delta} = \frac{X}{U},$$

$$\frac{1}{x-\gamma} + \frac{1}{x-\alpha} - \frac{1}{x-\beta} - \frac{1}{x-\delta} = \frac{Y}{U},$$

$$\frac{1}{x-\alpha} + \frac{1}{x-\beta} - \frac{1}{x-\gamma} - \frac{1}{x-\delta} = \frac{Z}{U},$$

therefore $$Y^2 - Z^2 = -4(\beta-\gamma)(\alpha-\delta)U,$$

or, if λ, μ, ν be roots of $4\rho^3 - I\rho + J = 0$,

$$Y^2 - Z^2 = 16(\mu-\nu)U,$$
$$Z^2 - X^2 = 16(\nu-\lambda)U,$$
$$X^2 - Y^2 = 16(\lambda-\mu)U.$$

Hence $$\Sigma(\mu-\nu)X^2 = 0,$$

$$\Sigma\lambda(\mu-\nu)X^2 = -16(\lambda-\mu)(\mu-\nu)(\nu-\lambda)U_x,$$

$$X^2+Y^2+Z^2 = -48H_x;$$

or, if X^2, Y^2, Z^2 be replaced by

$$16\xi^2(\lambda-\mu)(\lambda-\nu), \quad 16\eta^2(\mu-\nu)(\mu-\lambda), \quad 16\zeta^2(\nu-\lambda)(\nu-\mu),$$

these relations become $$\xi^2+\eta^2+\zeta^2 \equiv 0,$$

$$\lambda\xi^2+\mu\eta^2+\nu\zeta^2 \equiv U_x,$$

$$\lambda^2\xi^2+\mu^2\eta^2+\nu^2\zeta^2 \equiv -H_x.$$

The fact of the squares of the quadratics X, Y, Z being linearly connected ensures that they are mutually harmonic, and that each therefore is the Jacobian of the other two.

II. *Second Order.*—Suppose $y = \frac{V}{U}$, where V and U are quadratics,

then $$\frac{dy}{\sqrt{y-\alpha'.y-\beta'.y-\gamma'.y-\delta'}} = \frac{p\,dx}{\sqrt{x-\alpha.x-\beta.x-\gamma.x-\delta}},$$

therefore $V-\alpha'U$, $V-\beta'U$ are squares, and

$$V-\gamma'U \equiv l(x-\alpha)(x-\beta),$$

$$V-\delta'U \equiv m(x-\gamma)(x-\delta).$$

Hence the most general transformation of the second order is

$$y = \frac{lX+mY}{l'X+m'Y} \text{ or } \frac{lY+mZ}{l'Y+m'Z} \text{ or } \frac{lZ+mX}{l'Z+m'X},$$

and all possible transformations must belong to one of these three classes.

Suppose we take $$z = \frac{Y}{Z},$$

then $$dz = \frac{J(Y,Z)}{Z^2}dx = p\frac{X}{Z^2}dx \quad \left[p = -4(\mu-\nu)\right],$$

therefore $$p\frac{dx}{\sqrt{U_x}} = \frac{Z^2dz}{X\sqrt{U_x}} = \frac{Z^2dz}{X\sqrt{Y^2-Z^2}}4\sqrt{\mu-\nu}$$

$$= 4\sqrt{\mu-\nu}\,\frac{dz}{\frac{X}{Z}\sqrt{z^2-1}}$$

$$= \frac{4(\mu-\nu)\,dz}{\sqrt{z^2-1}\,\sqrt{z^2(\lambda-\nu)-(\lambda-\mu)}},$$

therefore $$\frac{dx}{\sqrt{U_x}} = \frac{dz}{\sqrt{z^2-1}\,\sqrt{z^2(\lambda-\nu)-(\lambda-\mu)}}.$$

We see, therefore, that we can reduce any elliptic element to its canonical form by means of 18 different quadratic transformations, viz., by putting $z = \frac{Y}{Z}$, $\frac{Z}{X}$, or $\frac{X}{Y}$, and then take the six linear transformations referred to in (I.).

I shall now apply this method to

$$\frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}.$$

The sextic covariant is $x(1-kx^2)(1+kx^2)$.

Hence, taking $$y = \frac{(1+k)\,x}{1+kx^2},$$

we get $$\frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}} = \frac{1}{1+k} \frac{dy}{\sqrt{1-y^2 \,.\, 1-\frac{4k}{(1+k)^2}y^2}},$$

which has the same modulus as in Landen's transformation, and which has the additional advantage of reducing an elliptic integral much more quickly than by Landen's formula.

By putting $y = \sqrt{-1}\tan\psi$, and $x = \sqrt{-1}\tan\phi$, Landen's formula is obtained.

The six quadratic forms corresponding to the quadratic factors $(1+k)\,x$ and $1+kx^2$ can be at once written out; they are

$$z \text{ or } \frac{1}{z} = y, \quad z \text{ or } \frac{1}{z} = \frac{y-\sqrt{\frac{2\sqrt{k}}{1+k}}}{y+\sqrt{\frac{2\sqrt{k}}{1+k}}}, \text{ and } z \text{ or } \frac{1}{z} = \frac{y-\sqrt{-\frac{2\sqrt{k}}{1+k}}}{y+\sqrt{-\frac{2\sqrt{k}}{1+k}}};$$

the other twelve forms can be derived in a similar manner.

III. *Third Order.*—If, when $y = -\frac{V}{U}$ (where U and V are cubics),

$$\frac{dy}{\sqrt{y-\alpha' \,.\, y-\beta' \,.\, y-\gamma' \,.\, y-\delta'}} = \frac{p\,dx}{\sqrt{x-\alpha \,.\, x-\beta \,.\, x-\gamma \,.\, x-\delta}},$$

then $\alpha' U+V$, $\beta' U+V$, $\gamma' U+V$, $\delta' U+V$ must each contain a square factor.

Now, any two cubics can be reduced by linear transformation to be the first differentials of a quartic; we shall therefore take

$$y = -\frac{U_2}{U_1}, \quad \text{where} \quad U = (abcde)(x, 1)^4.$$

Since $\alpha' U_1+U_2$, $\beta' U_1+U_2$, $\gamma' U_1+U_2$, $\delta' U_1+U_2$ each contain a square factor, α', β', γ', δ' must be roots of that covariant which has for its source $a^2d^2-6abcd-3b^2c^2+4ac^3+4b^3d$, that is, of $U_y J-H_y I$. The problem therefore reduces to find the result of substituting

$$y = -\frac{U_2}{U_1} \text{ in } \frac{dy}{\sqrt{H_y I-U_y J}}.$$

As the result of substitution in $H_y I-U_y J$ is a covariant, I shall only find the leading term, b and d being each zero.

The leading term is

$$(ae+3c^2)\,ce \cdot a^4-(ace-c^3)\,ea^4 = 4a^4c^3e$$

$$= a^2c^2\left[l\,(ace-c^3)\,a+m\,(ae+3c^2)\,ac\right],$$

where
$$l+m = 4,$$
$$l-3m = 0,$$

therefore
$$l = 3, \quad m = 1.$$

Hence result of substitution in $H_y I-U_y J$ is

$$\frac{H_x^2\,(IH_x+3JU_x)}{U_1^4},$$

also
$$dy = \frac{3H_x}{U_1^2}\,dx,$$

therefore
$$\frac{dy}{\sqrt{H_y I-U_y J}} = \frac{3\,dx}{\sqrt{H_x I+3U_x J}}.$$

Now, instead of putting $y = -\dfrac{U_2}{U_1}$,

we may take
$$y = -\frac{\alpha U_2+\beta H_2}{\alpha U_1+\beta H_1},$$

and the last result will become

$$\frac{dy}{\sqrt{H_{a.\beta.y} I_{a.\beta} - U_{a.\beta.y} J_{a.\beta}}} = \frac{3\,dx}{\sqrt{H_{a.\beta.x} I_{a.\beta} + 3U_{a.\beta.x} J_{a.\beta}}},$$

where $$U_{a.\beta.x} \equiv \alpha U_x + \beta H_x,$$

$$H_{a.\beta.x} \equiv \text{Hessian of } \alpha U_x + \beta H_x,$$

$I_{a.\beta}$ and $J_{a.\beta}$ the invariants of $\alpha U + \beta H$;

or, expanding, $$\frac{dy}{\sqrt{AU_y + BH_y}} = \frac{3\,dx}{\sqrt{CU_x + DH_x}},$$

where $$A = \left(\frac{\alpha\beta}{6} I + \frac{\beta^2}{4} J\right) I_{a.\beta} - \alpha J_{a.\beta},$$

$$B = \left(\alpha^2 - \frac{\beta^2}{12} I\right) I_{a.\beta} - \beta J_{a.\beta},$$

$$C = \left(\frac{\alpha\beta}{6} I + \frac{\beta^2}{4} J\right) I_{a.\beta} + 3\alpha J_{a.\beta},$$

$$D = \left(\alpha^2 - \frac{\beta^2}{12} I\right) I_{a.\beta} + 3\beta J_{a.\beta},$$

$$I_{a.\beta} = \alpha^2 I + 3\alpha\beta J + \frac{\beta^2}{12} I^2,$$

$$J_{a.\beta} = \alpha^3 J + \alpha^2\beta \frac{I^2}{6} + \alpha\beta^2 \frac{IJ}{4} + \beta^3 \frac{54J^2 - I^3}{216}.$$

From the result obtained above, we shall derive a few special results.

If $D = 0$, then $\alpha : \beta$ is determined from the equation

$$48\alpha^4 I + 288\alpha^3\beta J + 24\alpha^2\beta^2 I^2 + 24\alpha\beta^3 IJ + \beta^4 (36J^2 - I^3) = 0,$$

an equation which possesses the same invariant property as Jacobi's modular equation $u^4 - v^4 + 2uv(1 - u^2v^2) = 0$,

viz., its "I" invariant vanishes.

By using any of the four roots of this equation in

$$y = -\frac{\alpha U_2 + \beta H_2}{\alpha U_1 + \beta H_1},$$

we get the four distinct transformations of the third order.

If $A : B = C : D$, the part under the radical is unaltered by the

substitution. The condition gives, for the determination of $\alpha : \beta$,

$$I_{\alpha.\beta} J_{\alpha.\beta} (4\alpha^3 - \alpha\beta^2 I - \beta^3 J) = 0.$$

The condition $4\alpha^3 - \alpha\beta^2 I - \beta^3 J = 0$ gives $\frac{\alpha}{\beta} = \lambda$, μ, or ν,

therefore $$\alpha U + \beta H = X^2,\ Y^2,\ \text{or}\ Z^2.$$

Hence $$y = -\frac{X_2}{X_1},\ -\frac{Y_2}{Y_1},\ \text{or}\ -\frac{Z_2}{Z_1},$$

as has been obtained already.

The case of $J_{\alpha.\beta} = 0$ reduces to

$$\frac{dy}{\sqrt{\lambda U_y - H_y}} = 3\frac{dx}{\sqrt{\lambda U_x - H_x}},$$

where $$4\lambda^3 - I\lambda + J = 0,$$

or $$\frac{dy}{X_y} = 3\frac{dx}{X_x},$$

with two other similar results, and may be neglected.

The case of $I_{\alpha.\beta} = 0$ gives a *bonâ fide* substitution in which

$$\frac{dy}{\sqrt{\alpha U_y + \beta H_y}} = \sqrt{-3}\,\frac{dx}{\sqrt{\alpha U_x + \beta H_x}}.$$

IV. *Fourth Order.*—If, when $y = \frac{V}{U}$ (where U and V are quartics),

$$\frac{dy}{\sqrt{y - \alpha'.y - \beta'.y - \gamma'.y - \delta'}} = \frac{J(U.V)\,dx}{\sqrt{V - \alpha' U.V - \beta' U.V - \gamma' U.V - \delta' U}}$$

$$= \frac{p\,dx}{\sqrt{x - \alpha.x - \beta.x - \gamma.x - \delta}},$$

then, since $J(U.V)$ is of the sixth degree, U and V must satisfy either of the following conditions:—

(1) Some three of the expressions $V - \alpha' U$, $V - \beta' U$, $V - \gamma' U$, $V - \delta' U$ are perfect squares.

Or, (2) Some two are perfect squares, and each of the remaining two contains a square factor.

Let us consider the first case. Assume

$$V-\alpha' U = P^2, \quad V-\beta' U = Q^2, \quad V-\gamma' U = R^2,$$

then
$$(\beta'-\gamma')\,P^2+(\gamma'-\alpha')\,Q^2+(\alpha'-\beta')\,R^2 \equiv 0.$$

Hence P, Q, R are quadratics mutually harmonic, and are consequently the harmonic factors of the sextic covariant of some quartic of which V and U must be quartic covariants.

Under these circumstances,

$$\frac{dy}{\sqrt{y-\alpha' . y-\beta' . y-\gamma' . y-\delta'}}$$

reduces to an element of the form

$$\frac{dx}{\sqrt{V-\delta' U}}.$$

Hence the most general transformation of the fourth order of $\frac{dx}{\sqrt{U_x}}$ is

$$y = \frac{\lambda U_x+\mu H_x}{\lambda' U_x+\mu' H_x}.$$

If we refer to the transformation of the second order, it is at once obvious that this last transformation is derivable from it as a result of a quadratic transformation of y. In fact, it is not difficult to show that the transformations (1) and (2) of this section are included in the form

$$y = \frac{lY^2+2mYZ+nZ^2}{l'Y^2+2m'YZ+n'Z^2},$$

where Y and Z are two of the harmonic factors of sextic covariant.

If we put
$$y = + \frac{H_x}{U_x},$$

we get
$$dy = \frac{2G_x}{U_x^2}\,dx,$$

where G_x is the sextic covariant, therefore

$$2\,\frac{dx}{\sqrt{U_x}} = \frac{U_x^{\frac{3}{2}}}{G_x}\,dy$$

$$= \frac{U_x^{\frac{3}{2}}\,dy}{\sqrt{4H_x^3-U_x^2 H_x I-U_x^3 J}},$$

therefore $$2\frac{dx}{\sqrt{U_x}} = \frac{dy}{\sqrt{4y^3 - Iy - J}}.$$

This transformation of the fourth order may also be obtained very naturally as follows.

The integral of $\frac{dx}{\sqrt{U_x}} + \frac{dy}{\sqrt{U_y}} = 0$ may be written in the following covariant form, viz.,

$$P(\rho U - H) - \left(\rho^2 - \frac{I}{12}\right)(x-y)^2 = 0,$$

or $$\rho = \frac{P(U) - \sqrt{U_x U_y}}{2(x-y)^2},$$

where ρ is an arbitrary constant.

$$H_x \equiv (b^2 - ac)x^4 + 2(bc - ad)x^3 + \ldots,$$

$$P(U) \equiv x^2(ay^2 + 2by + c) + 2x(by^2 + 2cy + d) + cy^2 + 2dy + e$$
$$= \text{middle emanant}.$$

On differentiating, we obtain

$$\frac{dx}{\sqrt{U_x}} + \frac{dy}{\sqrt{U_y}} = \frac{d\rho}{\sqrt{4\rho^3 - I\rho - J}}.$$

Let the initial values of x and y be z, then the initial value of ρ is

$$\frac{H_z}{U_z}.$$

Hence $$\int_z^x \frac{dx}{\sqrt{U_x}} + \int_z^y \frac{dy}{\sqrt{U_y}} = \int_{H_z/U_z}^{\rho} \frac{d\rho}{\sqrt{4\rho^3 - I\rho - J}}.$$

If $x = y$, we obtain

$$2\int_z^x \frac{dy}{\sqrt{U_x}} = \int_{H_z/U_z}^{H_x/U_x} \frac{d\rho}{\sqrt{4\rho^3 - I\rho - J}},$$

which is the transformation of the fourth order.

If δ be a root of the equation $U_x = 0$ or $U_y = 0$, and if we take δ as the initial value of x and y, we have

$$\int_x^{\delta} \frac{dx}{\sqrt{U_x}} + \int_y^{\delta} \frac{dy}{\sqrt{U_y}} = \int_{\rho}^{\infty} \frac{d\rho}{\sqrt{4\rho^3 - I\rho - J}}.$$

Hence, putting $y = \delta$ in the formula

$$\rho = \frac{P(U) - \sqrt{U_x U_y}}{2(x-y)^2},$$

we obtain

$$\rho = \frac{x^2(a\delta^2+2b\delta+c)+2x(b\delta^2+2c\delta+d)+c\delta^2+2d\delta+e}{2(x-\delta)^2},$$

the equivalent elliptic formula being

$$u = \int_x^i \frac{dx}{\sqrt{U_x}} = \int_\rho^\infty \frac{d\rho}{\sqrt{4\rho^3 - I\rho - J}}.$$

We thus obtain two well-known formulæ

$$p(u) = \frac{x^2(a\delta^2+2b\delta+c)+2x(b\delta^2+2c\delta+e)+c\delta^2+2d\delta+e}{2(x-\delta)^2},$$

and

$$p(2u) = \frac{H_x}{U_x},$$

where

$$u = \int_x^i \frac{dx}{\sqrt{U_x}} = \int_\rho^\infty \frac{d\rho}{\sqrt{4\rho^3 - I\rho - J}}.$$

As an example of this last method, I shall show how to find the value of

$$\frac{(lx^2+2mx+n)\,dx}{\sqrt{(aU_x+\beta H_x)(a'U_x+\beta' H_x)}}.$$

If λ, μ, ν be the roots of $4\rho^3 - I\rho + J = 0$,

then $\lambda U - H$, $\mu U - H$, $\nu U - H$ are perfect squares, therefore

$$lx^2+2mx+n \equiv p\sqrt{\lambda U - H} + q\sqrt{\mu U - H} + r\sqrt{\nu U - H}.$$

Now, put

$$y = -\frac{H_x}{U_x},$$

and we get

$$\frac{dy}{\sqrt{4y^3 - Iy - J}} = +2\frac{dx}{\sqrt{U_x}},$$

therefore

$$\frac{(lx^2+2mx+n)\,dx}{\sqrt{(aU_x+\beta H_x)(a'U_x+\beta' H_x)}}$$

$$= \frac{\sqrt{U_x}\,(p\sqrt{\lambda U_x - H_x} + q\sqrt{\mu U_x - H_x} + r\sqrt{\nu U_x - H_x})}{\sqrt{(aU_x+\beta H_x)(a'U_x+\beta' H_x)}} \cdot \frac{dy}{\sqrt{4y^3 - Iy - J}}$$

$$= \frac{p\sqrt{y-\lambda} + q\sqrt{y-\mu} + r\sqrt{y-\nu}}{\sqrt{(a-\beta y)(a'-\beta' y)}} \cdot \frac{dy}{\sqrt{y-\lambda \,.\, y-\mu \,.\, y-\nu}}$$

$$= \frac{dy}{\sqrt{(a-\beta y)(a'-\beta' y)}} \left(\frac{p}{\sqrt{y-\mu \,.\, y-\nu}} + \frac{q}{\sqrt{y-\nu \,.\, y-\lambda}} + \frac{r}{\sqrt{y-\lambda \,.\, y-\mu}} \right).$$

A particular case of this is the well-known hyper-elliptic integral which can be reduced to elliptic elements, viz.,

$$\frac{dx}{\sqrt{1-x^2 \cdot 1-k^2x^2 \cdot 1-k'^2x^2 \cdot 1-k^2k'^2x^2}}.$$

It being obvious that $(1-x^2)(1-k^2k'^2x^2)$ and $(1-k^2x^2)(1-k'^2x^2)$ are quartics derivable from the same fundamental quartic. In short, they have the same sextic covariants.

I hope to be able to continue the discussion of the elliptic transformations on the covariant basis in a future paper. I think, at any rate, it is very interesting to see how the transformations of the different orders are to be derived from the covariants in their most general forms.

Note on the Arc of a Sphero-Conic. *By* H. F. W. Burstall, St. John's College, Cambridge.

[*Read November 11th*, 1886.]

Let the sphere be of unit radius; the polar equation of a sphero-conic is

$$\sin^2\rho\left\{\sin^2\theta\left(1+\frac{1}{a^2}\right)+\cos^2\theta\left(1+\frac{1}{b^2}\right)\right\}=1.$$

If P is the pole of a great circle touching the sphero-conic, the locus of P is called the polar reciprocal of the first conic, and its equation is well known to be

$$\sin^2\rho\left\{\sin^2\theta\,(1+a^2)+\cos^2\theta\,(1+b^2)\right\}=1.$$

Let AQP be the arc of a sphero-conic; A being the end of the major axis, TA and TP the tangents at A and P, AP the chord of contact, and $A'P'$ the arc of the reciprocal.

We have A' the pole of the tangent at A, and as the point of contact moves round to P the pole moves from A' to P', and the chord $A'P'$ is the locus of the poles of all great circles passing through T. Thus the triangle formed by the two tangents and the arc of the

conic is reciprocal to the triangle formed by the arc $A'P'$ and the chord $A'P'$.

Let the area $OA'P'Q' = \Delta_1$ and the area $OA'P' = \Delta$, let $TP = t_1$, $TA = t_2$, arc $AP = s$, and $\angle A'OP' = \theta'$.

Then (Williamson's "Integral Calculus," Art. 188)

$$\Delta_1 - \Delta + s + t_1 + t_2 = 2\pi,$$

and, remembering that the sphere is of unit radius,

$$\Delta_1 = \int_0^{\theta'} (1 - \cos\rho)\, d\theta'$$

$$= \theta' - \int_0^{\theta'} \left\{ 1 - \frac{1}{\cos\theta'\,(1+a^2) + \sin\theta'\,(1+b^2)} \right\}^{\frac{1}{2}} d\theta',$$

therefore

$$s + t_1 + t_2 + \theta' - \Delta = 2\pi + \int_0^{\theta'} \left\{ \frac{a^2\cos^2\theta' + b^2\sin^2\theta'}{1 + a^2\cos^2\theta' + b^2\sin^2\theta'} \right\}^{\frac{1}{2}} d\theta'.$$

[In the integral

$$\int_0^{\psi} \left\{ \frac{a^2\cos^2\psi + b^2\sin^2\psi}{\cos^2\psi\,(1+a^2) + \sin^2\psi\,(1+b^2)} \right\}^{\frac{1}{2}} d\psi, \qquad a > b,$$

put

$$\frac{b}{a}\tan\psi = \tan\phi,$$

and it reduces to

$$\int_0^{\phi} \frac{a^2b^2}{\cos\phi\sqrt{b^2\,(1+a^2) + (1+b^2)\,a^2\tan^2\phi}} \, \frac{d\phi}{b^2\cos^2\phi + a^2\sin^2\phi}$$

$$= \int_0^{\phi} \frac{a^2b^2\,d\phi}{\left\{a^2 - (a^2-b^2)\cos^2\phi\right\}\sqrt{a^2(1+b^2) - \cos^2\phi\left\{a^2(1+b^2) - b^2(1+a^2)\right\}}}$$

$$= \frac{a^2b^2}{a^3\,(1+b^2)^{\frac{1}{2}}} \int_0^{\phi} \frac{d\phi}{\left(1 - \frac{a^2-b^2}{a^2}\cos^2\phi\right)\sqrt{1 - \frac{a^2-b^2}{a^2\,(1+b^2)}\cos^2\phi}},$$

which is an elliptic integral of the third kind. (See Chasles on "Spherical Conics," translated by Graves, Appendix, p. 111.]

In any spherical curve, let ON be the perpendicular from the pole on the tangent, and let $NP = t$, $ON = p$, $\angle NOA = \omega$.

We have

$$s_1 - t = \int_0^{\omega} \sin p\, d\omega\,;$$

where $s_1 = \text{arc}\, AP$. To find $\sin p$ in terms of ω; the tangent plane at

P is $$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = z_1 z.$$

The equation of a plane perpendicular to this, and containing the axis of z, is

$$\frac{x_1}{a^2} y - \frac{y_1}{b^2} x = 0,$$

therefore $$\tan \omega = \frac{y}{x} = \frac{y_1}{x_1} \cdot \frac{a^2}{b^2}.$$

Now, O is the point (0, 0, 1), therefore the perpendicular from O on the tangent plane is

$$\frac{z_1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + z_1^2}};$$

but $$z_1^2 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2},$$

therefore $$(\text{perpendicular})^2 = \frac{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}$$

$$= \frac{a^2 \cos^2 \omega + b^2 \sin^2 \omega}{1 + a^2 \cos^2 \omega + b^2 \sin^2 \omega}.$$

Now, $$p = \sin p,$$

therefore $$s_1 - t = \int_0^\omega \sqrt{\frac{a^2 \cos^2 \omega + b^2 \sin^2 \omega}{1 + a^2 \cos^2 \omega + b^2 \sin^2 \omega}}\, d\omega.$$

Making $\omega = \theta'$, we have

$$s + t_1 + t_2 + \theta' - \Delta = 2\pi + s_1 - t,$$

therefore $$s_1 - s = t + t_1 + t_2 + \theta' - \Delta - 2\pi.$$

Hence we have the analogue of Fagnani's theorem for a sphero-conic.

The Theory of a Multilinear Partial Differential Operator, with Applications to the Theories of Invariants and Reciprocants.

By P. A. MacMahon.

[*Read November 11th*, 1886.]

The following paper is an attempt to study the theories of invariants and reciprocants together, by considering them both to be members of one large family of algebraic forms.

Recent researches* have brought to light such extraordinary analogies between them, that no surprise is occasioned at the circumstance that a main cause is found to be that they share these similarities with an infinite number of other members of the same group.

It is, however, somewhat remarkable that pure reciprocants are not at once seen to belong to the family, but require that modification or straining of elements which it occurred to Sylvester to adopt; M. Halphen had indeed previously strained the elements of his differential invariants in exactly the same manner, but his only reason for so doing appears to have been that he thereby greatly simplified the numerical coefficients of the algebraic forms to which his processes led him. Sylvester, however, having discovered that Halphen's forms were but ordinary algebraic invariants in a slightly altered garb, went at once to the root of the matter, and, by straining the whole of his pure reciprocants, at once established an intimate bond of connection between the two theories.

It will be seen that the Theory of Invariants stands in a unique and fundamental position in the discussion, inasmuch as it is auxiliary to the theory of every other member of the family.

Most of the properties which are known to be common to the two allied theories to which this paper has special reference are found to be inherent in them, by reason of the multilinear forms of the partial differential equations which serve as their algebraic definitions.

Beyond this, each system may possess characteristic features unshared by any of the others; such, for instance, as the law of reciprocity, which occurs only in the theory of invariants.

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* The following memoirs and papers should be consulted:—

"Thèse sur les invariants différentiels," Halphen; Paris, 1878.

"Lectures on the Theory of Reciprocants," Sylvester; *American Journal of Mathematics*, Vol. VIII., No. 4 *et seq.* (now in progress).

Papers by Elliott, Hammond, Rogers, and Leudesdorf, in Vol. XVII. of the *Proceedings of the London Mathematical Society*.

The quantic, analogous to that which has been termed by Hammond and Leudesdorf a quasi-covariant, I have preferred to call the "auxiliary quantic," since it appears to me to correspond, in the theory of invariants, to that particular covariant which is itself the original fundamental quantic.

We know the relations existing between these auxiliary quantics, and the solutions of the partial differential equations of the theories of invariants and pure reciprocants, as well as their geometrical meaning; it will be an interesting subject of future enquiry to detect these relations and their significance in other cases.

Incidentally, many new results are obtained in both of the known theories, and doubtless the matter of this paper is susceptible of much further development, generally and particularly.

The whole point of the investigation is in the construction of the multilinear operator, which I now proceed to define.

I. *Definition of the Multilinear Operator.*

Taking the letters to be $(a_0, a_1, a_2, a_3, \ldots) = (a, b, c, d, \ldots)$, viz., writing the second letter either as b or as a_1, the third as c or as a_2, and so on, denote the multinomial expression

$$\Sigma \frac{(m-1)!}{\kappa_0!\,\kappa_1!\,\kappa_2!\,\kappa_3!\,\ldots} a^{\kappa_0} b^{\kappa_1} c^{\kappa_2} d^{\kappa_3} \ldots \begin{pmatrix} \kappa_0+\kappa_1+\kappa_2+\kappa_3+\ldots = m \\ \kappa_1+2\kappa_2+3\kappa_3+\ldots = s \end{pmatrix}$$

by $$A_{s,m};$$

the multilinear operator of four elements is

$$(\mu, \nu;\ m, n) \equiv \sum_{s=0}^{s=\infty} (\mu+s\nu)\, A_{s,m}\, \partial_{a_{n+s}};$$

this may be written symbolically in the form

$$\frac{1}{m} \{\mu\partial_{a_n},\ (\mu+\nu)\,\partial_{a_{n+1}},\ (\mu+2\nu)\,\partial_{a_{n+2}},\ \ldots\}\ (a, b, c, \ldots)^m,$$

where $$(a, b, c, \ldots)^m = (A, B, C, \ldots),$$

if $A, B, C, \ldots$ are the coefficients of the successive powers of x in the expansion of $(a+bx+cx^2+\ldots)^m$; viz.,

$$\begin{aligned} A+Bx+Cx^2+\ldots &= (a+bx+cx^2+\ldots)^m \\ &= a^m \\ &\quad + \frac{m}{1}\, a^{m-1}(bx+cx^2+\ldots) \\ &\quad + \frac{m\,.\,m-1}{1\,.\,2}\, a^{m-2}(bx+cx^2+\ldots)^2 \\ &\quad + \&c.\ldots; \end{aligned}$$

that is, $A = a^m$,

$$B = \frac{m}{1} a^{m-1}b,$$

$$C = \frac{m}{1} a^{m-1}c + \frac{m \cdot m-1}{1 \cdot 2} a^{m-2}b^2,$$

$$D = \frac{m}{1} a^{m-1}d + \frac{m \cdot m-1}{1 \cdot 2} a^{m-2}2bc + \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3} a^{m-3}b^3,$$

$$\ldots = \ldots\ldots,$$

$$\ldots = \ldots\ldots,$$

each term being obtained from the next preceding one by operating upon it according to Arbogast's rule of the last and the last but one.

The symbol thus denotes the operator

$$\frac{1}{m} \{\mu A\partial_{a_n} + (\mu+\nu)\, B\partial_{a_{n+1}} + (\mu+2\nu)\, C\partial_{a_{n+2}} + \ldots\},$$

or, at full length for a few terms,

$$(\mu, \nu;\ m, n) = \mu \frac{1}{m} a^m \partial_{a_n} + (\mu+\nu)\, a^{m-1}b\partial_{a_{n+1}}$$

$$+(\mu+2\nu) \left\{ a^{m-1}c + \frac{1}{2!}(m-1)\, a^{m-2}b^2 \right\} \partial_{a_{n+2}}$$

$$+(\mu+3\nu) \left\{ a^{m-1}d + (m-1)\, a^{m-2}bc + \frac{1}{3!}(m-1)(m-2)\, a^{m-3}b^3 \right\} \partial_{a_{n+3}}$$

$$+(\mu+4\nu) \left\{ a^{m-1}e + (m-1)\, a^{m-2}bd + \frac{1}{2!}(m-1)\, a^{m-2}c^2 \right.$$

$$+ \frac{1}{2!}(m-1)(m-2)\, a^{m-3}b^2c$$

$$\left. + \frac{1}{4!}(m-1)(m-2)(m-3)\, a^{m-4}b^4 \right\} \partial_{a_{n+4}}$$

$$+\ldots \quad \ldots \quad \ldots \quad \ldots$$

II. *Further Symbolical Representation.*

Mr. Hammond has made the very important observation, that the operator $(\mu, \nu;\ m, n)$ may be written in the symbolic form

$$\frac{1}{m} U^m [\mu\partial_{a_n}],$$

or as $$\int U^{m-1} dU \left[\mu \partial_{a_n}\right],$$

where $$U = a_0 + a_1 x + a_2 x^2 + \ldots,$$

and, after expanding $\frac{1}{m} U^m$ in ascending powers of x,

$$x^s \left[\mu \partial_{a_n}\right]$$

is to be interpreted as meaning

$$(\mu + s\nu)\, \partial_{a_{n+s}}.$$

The advantage of this compendious symbolism will frequently appear in what follows.*

III. *Preliminary Remarks on the Operator and Explanation of the Notation used.*

Of the four elements μ, ν, m, n, only the last need be integral; the first three may have general values. It should be remarked that, if ν has a non-zero value, it may be put equal to unity without loss of generality, since

$$(\mu, \nu;\ m, n) = \nu \left(\frac{\mu}{\nu},\ 1;\ m, n\right), \qquad (\nu \neq 0);$$

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* Mr. Hammond observes, in a communication to the author, dated Dec. 11th, 1886, that, in the limiting case when $m = 0$,

$$(0, 1;\ 0, -1)$$

represents the operator

$$-\log\left(1 + \frac{a_1}{a_0} x + \frac{a_2}{a_0} x^2 + \ldots\right)\left[0 \partial_{a-1}\right],$$

where, after expanding the logarithm, we write

$$x^s \left[0 \partial_{a-1}\right] = s \partial_{a_{s-1}}.$$

Thus $$(0, 1;\ 0, -1)$$

$$= \frac{a_1}{a_0} \partial_{a_0} + \frac{2a_0 a_2 - a_1^2}{a_0^2} \partial_{a_1} + \frac{3a_0^2 a_3 - 3a_0 a_1 a_3 + a_1^3}{a_0^3} \partial_{a_2} + \ldots,$$

the coefficients being the sums of the powers of the roots of the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots = 0, \qquad (n = \infty).$$

The alternant of $(0, 1;\ 0, -1)$, as above written, and $(4, 1;\ 2, 1)$ is

$$2\,(a_0 \partial_{a_0} + a_1 \partial_{a_1} + a_2 \partial_{a_2} + \ldots),$$

so that $(0, 1;\ 0, -1)$ is a reversor to Sylvester's $V\{\equiv (4, 1;\ 2, 1)\}$.

it is, however, convenient in the general investigation to retain ν, so that its value may be unrestricted; in particular cases which arise when the zero-value is obviously excluded, ν will be put equal to unity.

In the theory of invariants there exists the seminvariant annihilator

$$(1, 1; 1, 1) \equiv a\partial_b + 2b\partial_c + 3c\partial_d + \ldots,$$

whilst, in the pure reciprocant theory, the annihilator is

$$(4, 1; 2, 1) \equiv 4 \,.\, \tfrac{1}{2}a^2\partial_b + 5ab\partial_c + 6\,(ac + \tfrac{1}{2}b^2)\,\partial_d + \ldots;$$

the present investigation embraces, therefore, both of these great theories, regarding each system of forms as a member of one enormous family cluster of algebraic forms.

It is proper to note some further particular cases :—viz.,

The annihilator of non-unitary symmetric functions of the roots (*i.e.*, symmetric functions expressible in terms of the sums of powers $s_2, s_3, s_4, \ldots, s_i$ not appearing) of the equation

$$x^n - bx^{n-1} + cx^{n-2} - \ldots = 0,$$

$$(n = \infty),$$

is $$(1, 0; 1, 1) \equiv a\partial_b + b\partial_c + c\partial_d + \ldots .$$

Also, Euler's operator is

$$(1, 0; 1, 0) \equiv a\partial_a + b\partial_b + c\partial_c + \ldots,$$

and the isobaric operator is

$$(0, 1; 1, 0) \equiv b\partial_b + 2c\partial_c + 3d\partial_d + \ldots .$$

I follow Sylvester in writing

$$PQ = P \,.\, Q + P * Q,$$

wherein, P and Q denoting operators, PQ denotes the successive operation of Q and P upon the operand; or, leaving out consideration of the operand, PQ denotes the complete performance of the operator P upon the operator Q; $P \,.\, Q$ denotes the operator resulting from the symbolic algebraic multiplication of the operators P and Q; $P * Q$ indicates the performance of the operator P as a pure differential operation upon the operator Q considered as an explicit function of the elements $a, b, c, \ldots$.

Adopting a suggestion made to me by Mr. J. Hammond, I further write $$PQ - QP = P * Q - Q * P$$ in the convenient form $$[P \,.\, Q].$$

IV. *The Multiplication of two Multilinear Operators.*

We have $(\mu', \nu'; m', n')(\mu, \nu; m, n)$

$= (\mu', \nu'; m', n') . (\mu, \nu; m, n) + (\mu', \nu'; m', n') * (\mu, \nu; m, n)$,

in the notation just explained.

Since $$\partial_a * \frac{U^m}{m}\left[\mu\partial_{a_n}\right] = U^{m-1}x^s\left[\mu\partial_{a_n}\right],$$

$$= U^{m-1}\left[(\mu+s\nu)\,\partial_{a_{n+s}}\right],$$

there results

$$\partial_{a_s} * (\mu, \nu; m, n) = (m-1)(\mu+s\nu, \nu; m-1, n+s) \quad \ldots\ldots(1).$$

Also $$\partial_{a_t} U^m = m\Sigma\partial_{a_t} A_{s,m}\, x^s = mU^{m-1}x^t,$$

therefore $$\Sigma\partial_{a_t} A_{s,m}\, x^s = (m-1)\left\{\Sigma A_{s,m-1}\, x^s\right\} x^t;$$

whence, by comparison of coefficients,

$$\partial_{a_t} A_{s,m} = (m-1)\, A_{s-t,m-1} \quad (s \geqq t) \quad \ldots\ldots\ldots\ldots(2).$$

We now find

$$(\mu', \nu'; m', n')\, A_{s,m} = \frac{m-1}{m'}\, U^{m'}\left[\mu'\partial_{a_{n'}}\right] A_{s,m}$$

$$= \frac{m-1}{m'}\, U^{m'}\left[\mu' A_{s-n',m-1}\right] \quad (s \geqq n'),$$

wherein, after expanding $U^{m'}$, we must write

$$x^t\left[\mu' A_{s-n',\,m-1}\right] = (\mu'+t\nu')\, A_{s-n'-t,m-1} \quad (s \geqq n'+t).$$

Consequently,

$$(\mu', \nu'; m', n')\, A_{s,m} = (m-1) \sum_{t=0}^{t=s-n'} (\mu'+t\nu')\, A_{t,m'}\, A_{s-n'-t,\,m-1}.$$

Now $$\sum_{t=0}^{t=\infty} A_{t,m'}\, x^t = \frac{1}{m'}\, U^{m'},$$

$$\sum_{\lambda=0}^{\lambda=\infty} A_{\lambda,m-1}\, x^\lambda = \frac{1}{m-1}\, U^{m-1},$$

and $$\sum_{t=0}^{t=\infty} tA_{t,m'}\, x^{t-1} = U^{m'-1}\frac{dU}{dx};$$

hence $$\sum_{t=0}^{t=s-n'} A_{t,m'} A_{s-n'-t,m-1} = \text{co. } x^{s-n'} \text{ in } \frac{1}{m'(m-1)} U^{m'+m-1}$$

$$= \frac{m'+m-1}{m'(m-1)} A_{s-n',\, m'+m-1},$$

and

$$\sum_{t=0}^{t=s-n'} t A_{t,m'} A_{s-n'-t,m-1} = \text{co. } x^{s-n'-1} \text{ in } \frac{1}{m-1} U^{m'+m-2} \frac{dU}{dx}$$

$$= \text{co. } x^{s-n'-1} \text{ in } \frac{1}{(m'+m-1)(m-1)} \frac{d}{dx} U^{m'+m-1}$$

$$= \frac{s-n'}{m-1} A_{s-n',\, m'+m-1}.$$

From the two identities,

$$\sum_{t=0}^{t=s-n'} A_{t,m'} A_{s-n'-t,m-1} = \frac{m'+m-1}{m'(m-1)} A_{s-n',\, m'+m-1},$$

$$\sum_{t=0}^{t=s-n'} t A_{t,m'} A_{s-n'-t,m-1} = \frac{s-n'}{m-1} A_{s-n',\, m'+m-1},$$

combined with a previous result, viz.,

$$(\mu', \nu';\, m', n')\, A_{s,m} = (m-1) \sum_{t=0}^{t=s-n'} (\mu'+t\nu')\, A_{t,m'} A_{s-n'-t,m-1},$$

we immediately obtain

$$(\mu', \nu';\, m', n')\, A_{s,m} = \left\{ (m'+m-1)\frac{\mu'}{m'} + (s-n')\nu' \right\} A_{s-n',\, m'+m-1} \quad \ldots(3).$$

Since $s \geqq n'$, put $s = n'+\kappa$, $\quad (\kappa = 0, 1, 2, \ldots \textit{ad inf.})$,

and then

$$(\mu', \nu';\, m', n')\, A_{n'+\kappa,\, m} = \left\{ (m'+m-1)\frac{\mu'}{m'} + \kappa\nu' \right\} A_{\kappa,\, m'+m-1},$$

we now reach the result

$$(\mu', \nu';\, m', n') * (\mu, \nu;\, m, n)$$

$$= \sum_{\kappa=0}^{\kappa=\infty} \left\{ (\mu', \nu';\, m', n')\, A_{n'+\kappa,m} \right\} \left\{ \mu + (n'+\kappa)\nu \right\} \partial_{a_{n'+n+\kappa}}$$

$$= \sum_{\kappa=0}^{\kappa=\infty} \left\{ (m'+m-1)\frac{\mu'}{m'} + \kappa\nu' \right\} \left\{ \mu + (n'+\kappa)\nu \right\} A_{\kappa;\, m'+m-1}\, \partial_{a_{n'+n+\kappa}}.$$

Observe that the expression just written down is a multilinear operator of six elements, and that the multiplication theorem thus is

$$(\mu', \nu';\ m', n')\ (\mu, \nu;\ m, n)$$

$$= (\mu', \nu';\ m', n') \,.\, (\mu, \nu;\ m, n)$$

$$+ \left\{ (m'+m-1)\frac{\mu'}{m'},\ \nu';\ \mu+n'\nu,\ \nu;\ m'+m-1,\ n'+n \right\} \quad \ldots\ldots(4).$$

Subtracting from (4) the similar equation obtained by an interchange of accented and unaccented letters, we find

$$(\mu', \nu';\ m', n')\ (\mu, \nu;\ m, n) - (\mu, \nu;\ m, n)\ (\mu', \nu';\ m', n')$$

$$= \sum_{\kappa=0}^{\kappa=\infty} \left[\left\{ (m'+m-1)\frac{\mu'}{m'} + \kappa\nu' \right\} \left\{ \mu + (n'+\kappa)\,\nu \right\} \right.$$

$$\left. - \left\{ (m'+m-1)\frac{\mu}{m} + \kappa\nu \right\} \left\{ \mu' + (n+\kappa)\,\nu' \right\} \right] A_{\kappa, m'+m-1}\, \partial_{a_{n'+n+\kappa}},$$

or

$$\left[(\mu', \nu';\ m', n') \,.\, (\mu, \nu;\ m, n) \right] = \sum_{\kappa=0}^{\kappa=\infty} (\mu_1 + \kappa\nu_1)\, A_{\kappa, m'+m-1}\, \partial_{a_{n'+n+\kappa}},$$

or

$$\left[(\mu', \nu';\ m', n') \,.\, (\mu, \nu;\ m, n) \right] = (\mu_1\, \nu_1;\ m'+m-1, n'+n),$$

where

$$\mu_1 = (m'+m-1) \left\{ \frac{\mu'}{m'}(\mu + n'\nu) - \frac{\mu}{m}(\mu' + n\nu') \right\},$$

$$\nu_1 = (n'-n)\,\nu'\nu + \frac{m-1}{m'}\,\mu'\nu - \frac{m'-1}{m}\,\mu\nu' \qquad \ldots\ldots(5).$$

Whence the fundamental theorem :—

"The alternant of two multilinear operators, of four elements each, is a multilinear operator of four elements."

This conclusion is the source of the succeeding developments which flow naturally from it.*

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* I had obtained a direct algebraic proof of this theorem, similar in essence to that given here, when Mr. Hammond pointed out that the free use of his symbolism would abbreviate the expressions involved, and thus render the investigation more compendious, and therefore bring it more under the eye of the reader. I accordingly adopted his suggestion.

The theorem may be also written in the following form, which is sometimes useful, viz.,

$$\left.\begin{aligned}
&\left[(\mu_2, \nu_2;\ m_2, n_2)\,.\,(\mu, \nu;\ m, n)\right] = M\,(\mu'', \nu'';\ m'', n''),\\
&\text{where}\\
&\mu_2 = (m''-m+1)\left\{m''\nu''\mu n+\mu''m\nu\ (n''-2n)-\mu''(m''-m)\,\mu\right\},\\
&\nu_2 = m''\nu''\ \left\{(n''-n)\,m\nu-(m''-2m+1)\,\mu\right\},\\
&m_2 = m''-m+1,\\
&n_2 = n''-n,\\
&M = \frac{m''}{m}\left\{m\nu\,(n''-2n)-(m''-m)\,\mu\right\}\left\{(n''-n)\,m\nu-(m''-2m+1)\,\mu\right\}\\
&\qquad\qquad +m''\,(m-1)\,\mu\nu n
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6).$$

Showing that in general ($n'' \lessgtr n$) an operator may be found whose alternant, with any one given operator, is any second given operator.

Denoting the alternant relation given above by

$$[P\,.\,Q] = R,$$

and any operand by F, the following conclusions are drawn:—

(i.) If $PF = QF = 0$, then $RF = 0$;

(ii.) If $QF = QPF = 0$,

or if $PF = PQF = 0$, then $RF = 0$;

(iii.) If $PQF = QPF = 0$, then $RF = 0$;

(iv.) If $RF = 0$ and $QF = 0$, or $PQF = 0$,

then either $PF = 0$ or $QPF = 0$.

V.—*Applications of Section IV.*

In the theory of invariants, we may write down the relations,

$$m'\left[(\mu', \nu';\ m', n')\,.\,(1, 1;\ 1, 1)\right]$$
$$= \left\{(n'-m'+1)\,\mu'-m'\nu',\ (n'-m')\,\nu';\ m', n'+1\right\} \quad \ldots\ldots\ldots(7),$$

$$\left[\left\{(n''-m''-1)\,\mu''+m''\nu'',\ (n''-m'')\,\nu'';\ m'',\ n''-1\right\}.(1,1;\ 1,1)\right]$$
$$=(n''-m'')\,(n''-m''-1)\,(\mu'',\ \nu'';\ m'',\ n'')\ \ldots\ldots\ldots\ldots(8);$$

and, in the theory of pure reciprocants,

$$m'\left[(\mu',\ \nu';\ m',\ n').(4,1;\ 2,1)\right]$$
$$=\left[(m'+1)\left\{(n'-2m'+4)\,\mu'-2m'\nu'\right\},\ \mu'+(n'-2m'+1)\,m'\nu';\right.$$
$$\left.m'+1,\ n'+1\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$\left[(\mu_2,\ \nu_2;\ m''-1,\ n''-1).(4,1;\ 2,1)\right]$$
$$=m''\,(n''-2m''+4)\,(n''-2m''+3)\,(\mu'',\ \nu'';\ m'',\ n'')\ \ldots\ldots(10),$$

where $\mu_2=(m''-1)\left\{(n''-2m''+2)\,\mu''+2m''\nu''\right\}$,

$\nu_2=(n''-2m''+5)\,m''\nu''-\mu''$;

also $$\left[(\mu_3,\ \nu_3;\ 3-m,\ 1-n).(\mu,\ \nu;\ m,\ n)\right]$$
$$=M_3\,(4,1;\ 2,1)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where $\mu_3=(3-m)\left\{4m\nu\,(1-2n)+2\,(n+2m-4)\,\mu\right\}$,

$\nu_3=2m\nu\,(1-n)-2\mu\,(3-2m)-m\,(m-1)\,\mu\nu n$,

$$M_3=\frac{2}{m}\left\{m\nu\,(1-2n)+(m-2)\,\mu\right\}\left\{m\nu\,(1-n)+(2m-3)\,\mu\right\}$$
$$+2\,(m-1)\,\mu\nu n.$$

From (5) or (7) is deduced

$$(1,1;\ 1,1).(4,1;\ 2,1)\big]=(2,1;\ 2,2)\ \ldots\ldots\ldots(12),$$

which proves that every projective reciprocant* is annihilated by the operation $(2,1;\ 2,2)$; and, further, that as regards all solutions of the equation $(2,1;\ 2,2)=0$, $(1,1;\ 1,1)$ and $(4,1;\ 2,1)$ are commutative operations.

This result may also be verified by (8), (9), and (10).

In (11), putting $(\mu,\nu;\ m,n)=(1,1;\ 1,1)$, we find

$$\left[(6,1;\ 2,0).(1,1;\ 1,1)\right]=-2\,(4,1;\ 2,1)\ \ldots\ldots\ldots(13),$$

indicating that the operation

$$(6,1;\ 2,0)=6\,\frac{a^2}{2}\,\partial_a+7ab\partial_b+8\left(ac+\frac{b^2}{2}\right)\partial_c+9\,(ad+bc)\,\partial_d+\ldots,$$

---

* These are M. Halphen's Differential Invariants ("Thèse sur les Invariants Différentiels," Paris, 1878); they have since been termed "Principiants" by Professor Sylvester (*American Journal of Mathematics*, Vol. ix., No. 1).

when performed upon a projective reciprocant, has the effect of converting it into another seminvariant of the same weight, and of degree higher by unity; thus

$$(6, 1; 2, 0)(a^2d-3abc+2b^3) = 15a\,(a^2d-3abc+2b^3);$$

in this particular case, the seminvariant is necessarily a factor of its altered self, but this is not so in general.

VI. *The conditions for the vanishing of the alternant.*

In order that the general alternant (5) may vanish, we must have

$$\frac{\mu'}{m'}(\mu+n'\nu)-\frac{\mu}{m}(\mu'+n\nu') = 0,$$

$$(n'-n)\nu'\nu+\frac{m-1}{m'}\mu'\nu-\frac{m'-1}{m}\mu\nu' = 0,$$

or, eliminating μ' and ν',

$$\{(n'-n)\,m\nu-(m'-1)\,\mu\}\{m(\mu+n'\nu)-m'\mu\}+m(m-1)\,\mu\nu n = 0,$$

which may be written

$$(m\nu n'-\mu m'+\mu)(m\nu n'-\mu m'+m\mu-m\nu n) = 0.$$

Firstly, we may have

$$m\nu n'-\mu m'+\mu = 0,$$

or
$$n' = \frac{\mu}{m\nu}(m'-1), \quad m' = 1+\frac{m\nu}{\mu}n';$$

and in this case

$$\mu' = \frac{m'n}{m-1}\nu' = \frac{n}{m-1}\left(1+\frac{m\nu}{\mu}n'\right)\nu'.$$

Secondly, if
$$m\nu n'-\mu m'+m\mu-m\nu n = 0,$$

then
$$n' = n+\frac{\mu}{m\nu}(m'-m), \quad m' = m+\frac{m\nu}{\mu}(n'-n),$$

and
$$\mu' = \frac{\mu}{m\nu}m'\nu' = \left\{\frac{\mu}{\nu}+m(n'-n)\right\}\nu'.$$

In these two cases, the zero value of ν' is obviously excluded, so that we may, without loss of generality, assign to ν' any convenient

values. Put, therefore, ν' equal to $m-1$ in the first case, and equal to $m\nu$ in the second; we thus obtain the two relations,

$$\left[\left\{nm', m-1;\ m',\ \frac{\mu}{m\nu}(m'-1)\right\}.(\mu,\nu;\ m,n)\right] = 0.....(14),$$

$$\left[\left\{\mu m', m\nu;\ m',\ n+\frac{\mu}{m\nu}(m'-m)\right\}.(\mu,\nu;\ m,n)\right] = 0.....(15).$$

In (14) $\frac{\mu}{m\nu}(m'-1)$ must be integral, whilst $\frac{\mu}{m\nu}(m'-m)$ must be so in (15).

In (14) and (15), we may make n' the independent variable, and obtain

$$\left[\left\{n\left(1+\frac{m\nu}{\mu}n'\right),\ m-1;\ 1+\frac{m\nu}{\mu}n',\ n'\right\}.(\mu,\nu;\ m,n)\right] = 0...(16),$$

$$\left[\left\{1+\frac{m\nu}{\mu}(n'-n),\ \frac{\nu}{\mu};\ \ m+\frac{m\nu}{\mu}(n'-n),\ n'\right\}.(\mu,\nu;\ m,n)\right] = 0...(17),$$

in which now only n' is restricted to be integral.

Denoting by $F(\mu,\nu;\ m,n)$ any homogeneous isobaric function which satisfies the multilinear partial differential equation

$$(\mu,\nu;\ m,n) = 0,$$

we draw the following inferences :—

(i.) If $\frac{\mu}{m\nu}(m'-1)$ be integral,

$$\left\{nm', m-1;\ m',\ \frac{\mu}{m\nu}(m'-1)\right\}F(\mu,\nu;\ m,n) = F(\mu,\nu;\ m,n),$$

and $\quad (\mu,\nu;\ m,n)\,F\left\{nm', m-1;\ m',\ \frac{\mu}{m\nu}(m'-1)\right\}$

$$= F\left\{nm', m-1;\ m',\ \frac{\mu}{m\nu}(m'-1)\right\}.$$

(ii.) If $\frac{\mu}{m\nu}(m'-m)$ be integral,

$$\left\{\mu m', m\nu;\ m', n+\frac{\mu}{m\nu}(m'-m)\right\}F(\mu,\nu;\ m,n) = F(\mu,\nu;\ m,n),$$

and $$(\mu, \nu\,;\, m, n)\, F\left\{\mu m', m\nu\,;\, m',\ n+\frac{\mu}{m\nu}(m'-m)\right\}$$

$$= F\left\{\mu m', m\nu\,;\, m',\ n+\frac{\mu}{m\nu}(m'-m)\right\}.$$

These two operators may be called, for the present, commutors of the operator $(\mu, \nu\,;\, m, n)$, and are of great importance in its theory.

VII. *Applications of Section VI.*

From (14) and (15) are deduced the relations,

$$\left[(m', 0\,;\, m', m'-1)\,.\,(1, 1\,;\, 1, 1)\right] = 0 \quad \ldots\ldots\ldots\ldots (18),$$

$$\left[(m', 1\,;\, m', m')\,.\,(1, 1\,;\, 1, 1)\right] = 0 \quad \ldots\ldots\ldots\ldots (19),$$

$$\left[(m', 1\,;\, m', 2m'-2)\,.\,(4, 1\,;\, 2, 1)\right] = 0 \quad \ldots\ldots\ldots\ldots (20),$$

$$\left[(2m', 1\,;\, m', 2m'-3)\,.\,(4, 1\,;\, 2, 1)\right] = 0 \quad \ldots\ldots\ldots\ldots (21).$$

In these relations it seems best to consider that, as a general rule, the fourth element is to be a positive (including zero) integer; but as to this there are possible exceptions (see *post*). Of course m' must be integral.

Whence, in the theory of invariants, we have the pair of operators

$$(m', 0\,;\, m', m'-1),$$

$$(m', 1\,;\, m', m'),$$

which convert seminvariants into other seminvariants; whilst, in the theory of pure reciprocants, we have the pair

$$(m', 1\,;\, m', 2m'-2),$$

$$(2m', 1\,;\, m', 2m'-3),$$

which convert pure reciprocants into other pure reciprocants.

It is convenient to write down the four series:—

I.

$$(1, 0\,;\, 1, 0) = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots$$

$$(1, 0\,;\, 2, 1) = \tfrac{1}{2}a^2\partial_b + ab\partial_c + (ac+\tfrac{1}{2}b^2)\,\partial_d + (ad+bc)\,\partial_e + \ldots$$

$$(1, 0\,;\, 3, 2) = \tfrac{1}{3}a^3\partial_c + a^2b\partial_d + (a^2c+ab^2)\,\partial_e + (a^2d+2abc+\tfrac{1}{3}b^3)\,\partial_f + \ldots$$

$$(1, 0\,;\, 4, 3) = \tfrac{1}{4}a^4\partial_d + a^3b\partial_e + (a^3c+\tfrac{3}{2}a^2b^2)\,\partial_f + (a^3d+3a^2bc+ab^3)\,\partial_g + \ldots$$

... ...

II.

$$(1, 1; 1, 1) = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$$

$$(2, 1; 2, 2) = 2 \cdot \tfrac{1}{2}a^2\partial_c + 3ab\partial_d + 4\,(ac + \tfrac{1}{2}b^2)\,\partial_e + 5\,(ad + bc)\,\partial_f + \ldots$$

$$(3, 1; 3, 3) = 3 \cdot \tfrac{1}{3}a^3\partial_d + 4a^2b\partial_e + 5\,(a^2c + ab^2)\,\partial_f + 6\,(a^2d + 2abc + \tfrac{1}{3}b^3)\,\partial_g + \ldots$$

$$(4, 1; 4, 4) = 4 \cdot \tfrac{1}{4}a^4\partial_e + 5a^3b\partial_f + 6\,(a^3c + \tfrac{3}{2}a^2b^2)\,\partial_g + 7\,(a^3d + 3a^2bc + ab^3)\,\partial_h + \ldots$$

... ...

III.

$$(1, 1; 1, 0) = a\partial_a + 2b\partial_b + 3c\partial_c + 4d\partial_d + \ldots$$

$$(2, 1; 2, 2) = 2 \cdot \tfrac{1}{2}a^2\partial_c + 3ab\partial_d + 4\,(ac + \tfrac{1}{2}b^2)\,\partial_e + 5\,(ad + bc)\,\partial_f + \ldots$$

$$(3, 1; 3, 4) = 3 \cdot \tfrac{1}{3}a^3\partial_e + 4a^2b\partial_f + 5\,(a^2c + ab^2)\,\partial_g + 6\,(a^2d + 2abc + \tfrac{1}{3}b^3)\,\partial_h + \ldots$$

$$(4, 1; 4, 6) = 4 \cdot \tfrac{1}{4}a^4\partial_g + 5a^3b\partial_h + 6\,(a^3c + \tfrac{3}{2}a^2b^2)\,\partial_i + 7\,(a^3d + 3a^2bc + ab^3)\,\partial_j + \ldots$$

... ...

IV.

$$(4, 1; 2, 1) = 4 \cdot \tfrac{1}{2}a^3\partial_b + 5ab\partial_c + 6\,(ac + \tfrac{1}{2}b^2)\,\partial_d + \ldots$$

$$(6, 1; 3, 3) = 6 \cdot \tfrac{1}{3}a^3\partial_d + 7a^2b\partial_e + 8\,(a^2c + ab^2)\,\partial_f + 9\,(a^3d + 2abc + \tfrac{1}{3}b^3)\,\partial_g + \ldots$$

$$(8, 1; 4, 5) = 8 \cdot \tfrac{1}{4}a^4\partial_f + 9a^3b\partial_g + 10\,(a^3c + \tfrac{3}{2}a^2b^2)\,\partial_h + 11\,(a^3d + 3a^2bc + ab^3)\,\partial_i + \ldots$$

$$(10, 1; 5, 7) = 10 \cdot \tfrac{1}{5}a^5\partial_h + 11a^4b\partial_i + 12\,(a^4c + 2a^3b^2)\,\partial_j + 13\,(a^4d + 4a^3bc + 2a^2b^3)\,\partial_k + \ldots$$

... ...

Of the above operators (1, 0; 2, 1) is important, since, so far, it appears to be unique in the theory of invariants, in that it lowers the weight of a seminvariant by unity, and does not, in general, annihilate it; it thus enables us to pass from a seminvariant table of weight w, to a number of seminvariants of weight $w-1$.

The same operator also occurs in Mr. L. J. Rogers' theory of homographic mixed reciprocants.*

* Mr. L. J. Rogers on "Homographic and Circular Reciprocants," *Proceedings of the London Mathematical Society*, Vol. XVII., p. 229.

Using unmodified letters, he obtains an annihilator

$$t^2\partial_a + 3at\partial_b + (4bt + 3a^2)\,\partial_c + (5ct + 10ab)\,\partial_d + \ldots,$$

in which, putting $2!\,a$ for a,

$3!\,b$ for b,

and, generally, $(s+2)!\,a_s$ for a_s,

which is precisely the modification employed in the case of pure reciprocants, the operator becomes

$$\frac{t^2}{2}\partial_a + ta\,\partial_b + \left(tb + \frac{a^2}{2}\right)\partial_c + (tc + ab)\,\partial_d + \ldots,$$

which, *quâ* the elements $t, a, b, c, \ldots$, is the foregoing operator

$$(1, 0; 2, 1).$$

It will also be noticed that one single operator,

$$(2, 1; 2, 2),$$

presents itself in each pair of series. Remarking that this same operator has already presented itself as the alternant of the seminvariant and pure reciprocant annihilators, we are able to enunciate the following extraordinary theorem :—

"The operator

$$(2, 1; 2, 2) = 2 \cdot \tfrac{1}{2}a^2\partial_c + 3ab\,\partial_d + 4\,(ac + \tfrac{1}{2}b^2)\,\partial_e + \ldots$$

is such that it transforms every seminvariant into another seminvariant, and every pure reciprocant into another pure reciprocant, whilst every form which is at once a seminvariant and a pure reciprocant it annihilates."

The operator, however, annihilates many seminvariants and pure reciprocants which do not partake of both characters; for example, in the *American Journal of Mathematics* (Vol. IX., pp. 22, 23), Professor Sylvester gives a set of pure reciprocant elements $A, B, C, \ldots$, and also a set of invariantive elements $A_0, A_1, A_2, \ldots$, such that any invariant in regard to either set of elements is a projective reciprocant; now it will be found that

$$(2, 1; 2, 2)$$

necessarily annihilates every element of both sets, whereas we know

that, of these elements, only the Mongian, A or A_0, is itself a projective reciprocant.

The fact is that, in regard to the solutions of the equation

$$(2, 1; 2, 2) = 0,$$

there exist for weights, not less than 3, two distinct sets of protomorphs, the one set being all seminvariants, and the other set all pure reciprocants.

This operation thus appears to be the central point of convergence of the two theories; symmetrically placed betwixt them, and occupying ground common to both, it treats each with the strictest impartiality. It is easily seen that (2, 1; 2, 2) annihilates, further, some forms which are neither seminvariants nor pure reciprocants; *ex. gr.*, the element b is obviously such a form.

We have here met with the curious circumstance of the alternant of two operators being a commutor of each, and it is interesting to now enquire what operators possess a common commutor, and under what circumstances this commutor is also their alternant.

VIII. *Operators possessing a common commutor.*

Consider the two operators $(\mu_1, 1; m_1, n_1)$, $(\mu_2, 1; m_2, n_2)$.

The first has a pair of commutors,

$$\left\{ n_1 s,\ m_1 - 1;\ s,\ \frac{\mu_1}{m_1}(s-1) \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$\left\{ \mu_1 s,\ m_1;\ s,\ n_1 + \frac{\mu_1}{m_1}(s-m_1) \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta);$$

and the second also a pair,

$$\left\{ n_2 s,\ m_2 - 1;\ s,\ \frac{\mu_2}{m_2}(s-1) \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha'),$$

$$\left\{ \mu_2 s,\ m_2;\ s,\ n_2 + \frac{\mu_2}{m_2}(s-m_2) \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta');$$

for the equivalence of (α) and (β'), we have $(m_1 \neq 1)$

$$\frac{n_1}{m_1 - 1} = \frac{\mu_2}{m_2},$$

and

$$\frac{\mu_1}{m_1}(s-1) = n_2 + \frac{\mu_2}{m_2}(s-m_2);$$

whence
$$n_1 = \frac{\mu_2}{m_2}(m_1-1),$$
$$\mu_1 = \frac{m_1}{m_2(s-1)}\{m_2 n_2 + \mu_2(s-m_2)\};$$
that is to say, subject to the conditions
$$\begin{cases} m_1 \neq 1, \\ \frac{\mu}{m}(m_1-1) \text{ a positive integer,} \end{cases}$$
the operators
$$\left\{ \left\{ \frac{m_1}{m(s-1)}\left[mn+\mu(s-m)\right], 1;\ m_1, \frac{\mu}{m}(m_1-1) \right\} \text{.....(22),} \right.$$
$$\left. (\mu, 1;\ m, n), \right.$$
possess the common commutor
$$\left\{ \mu s, m;\ s, n+\frac{\mu}{m}(s-m) \right\} \text{.....(23).}$$

Thus in this case there is a doubly infinite set; but, if $m_1=1$, we have
$$n_1 = \mu_2,$$
$$m_2 = 0,$$
giving no result, as then the fourth element becomes ∞.

Again, for the equivalence of (β) and (α'), if $m_2 \neq 1$,
$$\frac{\mu_1}{m_1} = \frac{n_2}{m_2-1},$$
$$n_1 + \frac{\mu_1}{m_1}(s-m_1) = \frac{\mu_2}{m_2}(s-1),$$
leading, as before, to
$$\mu_1 = \frac{n_2}{m_2-1} m_1,$$
$$n_1 = \frac{\mu_2}{m_2}(s-1) - \frac{n_2}{m_2-1}(s-m_1);$$
whence, if $m_2 \neq 1$,
$$\frac{\mu_2}{m_2}(s-1) - \frac{n_2}{m_2-1}(s-m_1) \text{ a positive integer,}$$

the two operators

$$\left\{ \begin{matrix} \left\{ \frac{nm_1}{m-1}, 1\,;\ m_1, \frac{\mu}{m}(s-1) - \frac{n}{m-1}(s-m_1) \right\} \\ (\mu, 1\,;\ m, n), \end{matrix} \right\} \quad \ldots\ldots(24),$$

possess the common commutor

$$\left\{ ns, m-1\,;\ s, \frac{\mu}{m}(s-1) \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(25),$$

giving another doubly infinite set of operators associated with $(\mu, 1\,;\ m, n)$, and linked to it by the bond of a common commutor.

In all, there are thus two doubly infinite sets of operators possessing the property under examination.

IX. *Operators whose common commutor is also their alternant.*

Recalling the result (5) *ante*,

$$\left[(\mu', \nu'\,;\ m', n') \,.\, (\mu, \nu\,;\ m, n) \right]$$

$$= \frac{1}{m'm} (\mu_1, \nu_1\,;\ m'+m-1, n'+n),$$

where $\mu_1 = (m'+m-1)\left\{ m\mu'(\mu+n'\nu) - m'\mu(\mu'+n\nu') \right\},$

$$\nu_1 = m'm\nu'\nu(n'-n) + m(m-1)\mu'\nu - m'(m'-1)\mu\nu',$$

we might, from the general results of § VIII., at once proceed to find the conditions that the common commutor may be also the alternant. It will, however, be conducive of generality, and also of clearness of vision, if we first find the expression for the alternant of

$$(\mu', \nu'\,;\ m', n') \text{ and } (\mu_1, \nu_1\,;\ m'+m-1, n'+n).$$

We have $\left[(\mu', \nu'\,;\ m', n') \,.\, (\mu_1, \nu_1\,;\ m'+m-1, n'+n) \right]$

$$= \frac{1}{m'(m'+m-1)} (\mu_2, \nu_2\,;\ 2m'+m-2, 2n'+n),$$

where

$$\mu_2 = (2m'+m-2)\left\{ (m'+m-1)\mu'(\mu_1+\nu_1 n') - m'\mu_1(\mu'+\nu' n'+\nu' n) \right\},$$

$$\nu_2 = -m'(m'+m-1)\nu'\nu_1 n + (m'+m-1)(m'+m-2)\mu'\nu_1$$

$$-m'(m'-1)\mu_1\nu';$$

and, substituting the foregoing values of μ_1 and ν_1, we find, after a somewhat long reduction,

$$\mu_2 = (2m'+m-2)(m'+m-1)\{m'\nu'n-(m-1)\mu'\}$$
$$\times\{(m'-m)\mu'\mu+m'\mu\nu'(n'+n)-2m\mu'\nu n'\},$$
$$\nu_2 = (m'+m-1)\{m'\nu'n-(m-1)\mu'\}$$
$$\times\{2m'(m'-1)\mu\nu'-m(m'+m-2)\mu'\nu-m'm\nu'\nu(n'-n)\};$$

whence the result is

$$[(\mu', \nu';\ m', n').(\mu_1, \nu_1;\ m'+m-1, n'+n)]$$
$$= \frac{m'\nu'n-(m-1)\mu'}{m'}$$
$$\times[(2m'+m-2)\{(m'-m)\mu'\mu+m'\mu\nu'(n'+n)-2m\mu'\nu n'\},$$
$$\{2m'(m'-1)\mu\nu'-m(m'+m-2)\mu'\nu-m'm\nu'\nu(n'-n)\};$$
$$2m'+m-2,\ 2n'+n] \ldots\ldots (26).$$

Now, an interchange of accented with unaccented letters causes μ_1 and ν_1 merely to change sign, the other two elements of the alternant $[(\mu', \nu';\ m', n').(\mu, \nu;\ m, n)]$ remaining unaltered; whence also the relation

$$[(\mu, \nu;\ m, n).(\mu_1, \nu_1;\ m'+m-1, n'+n)]$$
$$= \frac{m\nu n'-(m'-1)\mu}{m}$$
$$\times[(2m'+m-2)\{(m'-m)\mu'\mu-m\mu'\nu(n'+n)+2m'\mu\nu'n\},$$
$$\{-2m(m-1)\mu'\nu+m'(m'+m-2)\mu\nu'-m'm\nu'\nu(n'-n)\};$$
$$2m'+m-2,\ 2n'+n] \ldots (27).$$

The two alternants on the left-hand sides of (26) and (27) will both vanish in four distinct cases.

Case 1.—If $m'\nu'n-(m-1)\mu' = 0,$
$m\nu n'-(m'-1)\mu = 0,$

from previous work it will be seen that ν' and ν cannot have both zero values, so that, putting

$$\nu' = 1,$$

we find $$\mu' = \frac{n}{m-1} m',$$

$$n' = \frac{\mu}{m\nu} (m'-1),$$

and the operator is

$$\left\{ \frac{n}{m-1} m', 1 ; m', \frac{\mu}{m\nu} (m'-1) \right\},$$

agreeing with (14).

Case 2.—Putting again $\nu' = 1$, we may have

$$(m'-m)\,\mu'\mu + m'\mu\,(n'+n) - 2m\mu'\nu n' = 0 \quad \ldots\ldots\text{(i.)},$$

$$2m'(m'-1)\,\mu - m\,(m'+m-2)\,\mu'\nu - m'm\nu\,(n'-n) = 0 \quad \ldots\ldots\text{(ii.)},$$

$$(m'-m)\,\mu'\mu - m\mu'\nu\,(n'+n) + 2m'\mu n = 0 \quad \ldots\ldots\text{(iii.)},$$

$$2m(m-1)\,\mu'\nu - m'\,(m'+m-2)\,\mu + m'm\nu\,(n'-n) = 0 \quad \ldots\ldots\text{(iv.)}.$$

From (i.) and (iii.), by subtraction,

$$(m'\mu - m\mu'\nu)(n'-n) = 0,$$

and from (ii.) and (iv.), by addition,

$$(m'\mu - m\mu'\nu)(m'-m) = 0;$$

of these two equations, the solutions

$$n'-n = 0,$$

$$m'-m = 0,$$

lead to $\mu' = \frac{\mu}{\nu}$, and the resulting operator is merely a particular case of that obtained from the solution

$$m'\mu - m\mu'\nu = 0,$$

which leads to $$\mu' = \frac{\mu}{m\nu} m',$$

and thence, from (i.), to

$$n' = n + \frac{\mu}{m\nu} (m'-m),$$

or the operator is

$$\left\{\frac{\mu m'}{m\nu}, 1;\ m',\ n+\frac{\mu}{m\nu}(m'-m)\right\},$$

agreeing with (15).

Case 3.—Again, we may have

$$(m'-m)\,\mu'\mu+m'\mu\nu'\,(n'+n)-2m\mu'\nu n'=0,$$

$$2m'\,(m'-1)\,\mu\nu'-m\,(m'+m-2)\,\mu'\nu-m'm\nu'\nu\,(n'-n)=0,$$

$$m\nu n'-(m'-1)\,\mu=0.$$

These three equations are not independent, being equivalent to but two; hence, taking the first and third of them together, we find

$$\mu'=\frac{m'\nu'}{m\nu}\cdot\frac{(m'-1)\,\mu+m\nu n}{m'+m-2},$$

$$n'=\frac{\mu}{m\nu}(m'-1),$$

results showing that the two operators

$$\left\{\frac{m'\nu'}{m\nu}\cdot\frac{(m'-1)\,\mu+m\nu n}{m'+m-2},\ \nu';\ m',\ \frac{\mu}{m\nu}(m'-1)\right\}\quad\ldots\ldots\ldots(28),$$

$$(\mu,\ \nu;\ m,\ n),$$

are such that their alternant is also one of their common commutors. On reference to the result marked (22), it will be seen that (putting $\nu'=\nu=1$, as we may) this new operator becomes identical with it on putting in the latter $s=m'+m-1$.

And thus, as a result, we find that the singly infinite set of operators represented by the triad

$$\left.\begin{array}{c}\left\{\dfrac{m'}{m}\cdot\dfrac{(m'-1)\,\mu+mn}{m'+m-2},\ 1;\ m',\ \dfrac{\mu}{m}(m'-1)\right\}\\ (\mu,\ 1;\ m,\ n)\\ \left\{(m'+m-1)\,\mu,\ m;\ m'+m-1,\ n+\dfrac{\mu}{m}(m'-1)\right\}\end{array}\right\}\quad\ldots\ldots(A)$$

is such that the third is the alternant of the first two, and at the same time a commutor of each.

Case 4.—Similarly, we may have

$$(m'-m)\,\mu'\mu - m\mu'\nu\,(n'+n) + 2m'\mu\nu' n = 0,$$

$$2m\,(m-1)\,\mu'\nu - m'\,(m'+m-2)\,\mu\nu' + m'm\,\nu'\nu\,(n'-n) = 0,$$

$$m'\nu' n - (m-1)\,\mu' = 0,$$

in which we may at once put $\nu' = \nu = 1$, and thence obtain

$$\mu' = \frac{n}{m-1}\,m',$$

$$n' = \frac{(m'+m-2)\,\mu - mn}{m},$$

and the operator thence written down, viz.,

$$\left\{\frac{n}{m-1}\,m',\ 1\ ;\ m',\ \frac{\mu}{m}\,(m'+m-2)-n\right\} \quad \ldots\ldots\ldots\ldots(29),$$

is seen to be identical with (24), when

$$s = m'+m-1.$$

As a consequence, we arrive at a second singly infinite set of triads of the type

$$\left.\begin{array}{c} \left\{\frac{n}{m-1}\,m',\ 1\ ;\ m',\ \frac{\mu}{m}\,(m'+m-2)-n\right\} \\ (\mu,\ 1\ ;\ m,\ n) \\ \left\{(m'+m-1)\,n,\ 1\ ;\ m'+m-1,\ \frac{\mu}{m}\,(m'+m-2)\right\} \end{array}\right\} \quad \ldots\ldots(B),$$

possessing the same properties as those marked (A).

X.—*Applications of Sections VIII. and IX.*

The two operators,

$$\left(\frac{m's}{s-1},\ 1\ ;\ m',\ m'-1\right) \quad \ldots\ldots\ldots\ldots(30),$$

$$(1,\ 1\ ;\ 1,\ 1),$$

possess the common commutor

$$(s,\ 1\ ;\ s,\ s) \quad \ldots\ldots\ldots\ldots(31).$$

Also the pair

$$\left\{\frac{m'\,(2s-3)}{s-1},\ 1\ ;\ m',\ 2\,(m'-1)\right\} \quad \ldots\ldots\ldots\ldots(32),$$

$$(4,\ 1\ ;\ 2,\ 1),$$

have a common commutor

$$(2s,\ 1;\ s,\ 2s-3) \quad \ldots\ldots(33);$$

and likewise the pair

$$(m',\ 1;\ m',\ m'+s-2) \quad \ldots\ldots(34),$$
$$(4,\ 1;\ 2,\ 1),$$

the common commutor

$$(s,\ 1;\ s,\ 2s-2) \quad \ldots\ldots(35).$$

Again, the three following triads of operators possess the same properties as the triads (A) and (B), viz.:—

$$\left.\begin{matrix} \left\{\frac{m'^2}{m'-1},\ 1;\ m',\ m'-1\right\} \\ (1,\ 1;\ 1,\ 1) \\ (m',\ 1;\ m',\ m') \end{matrix}\right\} \quad \ldots\ldots(36),$$

$$\left.\begin{matrix} (2m'-1,\ 1;\ m',\ 2m'-2) \\ (4,\ 1;\ 2,\ 1) \\ (2m'+2,\ 1;\ m'+1,\ 2m'-1) \end{matrix}\right\} \quad \ldots\ldots(37),$$

$$\left.\begin{matrix} (m',\ 1;\ m',\ 2m'-1) \\ (4,\ 1;\ 2,\ 1) \\ (m'+1,\ 1;\ m'+1,\ 2m') \end{matrix}\right\} \quad \ldots\ldots(38).$$

XI.—*Combination with the operation* ∂_{a_t}.

When $s \geqq t$ and $m > 1$,

$$\partial_{a_t} A_{s,m} = (m-1)\, A_{s-t,m-1},$$

and

$$\partial_{a_t} A_{s,1} = 0,$$

except when $t = s$, and then

$$\partial_{a_t} A_{t,1} = 1.$$

Bearing these modifications in mind, we have

$$(\mu,\ \nu;\ m,\ n)\, \partial_{a_t} = (\mu,\ \nu;\ m,\ n) \,.\, \partial_{a_t},$$

$$\partial_{a_t} (\mu,\ \nu;\ m,\ n) = \partial_{a_t} \,.\, (\mu,\ \nu;\ m,\ n) + (m-1) \sum_{s=t}^{s=\infty} (\mu+s\nu)\, A_{s-t,m-1}\, \partial_{a_{n+s}},$$

$$= (\mu,\ \nu;\ m,\ n) \,.\, \partial_{a_t} + (m-1)(\mu+t\nu,\ \nu;\ m-1,\ n+t),$$

whence

$$\left[(\mu, \nu;\ m, n)\,.\,\partial_{a_t}\right] = -(m-1)(\mu+t\nu, \nu;\ m-1, n+t);$$

or, if the operand F be a solution of

$$(\mu, \nu;\ m, n) = 0,$$

$$(\mu, \nu;\ m, n)\,\partial_{a_t} F = -(m-1)(\mu+t\nu, \nu;\ m-1, n+t)\,F,$$

put $t = j-s$, where j is the extent (*i.e.*, weight of most advanced letter) of the operand; then

$$(\mu, \nu;\ m, n)\,\partial_{a_{j-s}} F = -(m-1)(\mu+j\nu-s\nu, \nu;\ m-1, n+j-s)\,F \ldots (39),$$

and the dexter side will vanish if

$$n+j-s > j,$$

if $$s < n.$$

Thus $$(\mu, \nu;\ m, n)\partial_{a_{j-s}} F = 0, \quad (s < n),$$

proving that $(s<n)$

$$\partial_{a_{j-s}} F$$

is also a solution of the partial differential equation

$$(\mu, \nu;\ m, n) = 0.$$

Also, writing the operator on the dexter of (39) at full length, we find, by repeated operation of $(\mu, \nu;\ m, n)$, the following theorem,

$$(\mu, \nu;\ m, n)^{s-n+2}\,\partial_{a_{j-s}} F = 0, \quad (s>n-2),$$

showing that $$(\mu, \nu;\ m, n)^{s-n+1}\,\partial_{a_{j-s}} F$$

is also a solution of $$(\mu, \nu;\ m, n) = 0.$$

These results are generalisations of theorems already known in the theories of invariants and pure reciprocants.

XII. *The auxiliary quantic.*

The following results are applications to the case of the general multilinear operator, here under examination, of theorems already obtained independently by Hammond and Leudesdorf.

In this connection, see Leudesdorf, *Proc. Lond. Math. Soc.*, Vol. XVII., p. 209, and Hammond in "Lectures on the Theory of Recipro-

cants," by J. J. Sylvester, in the forthcoming Vol. IX., No. 2, of the *American Journal of Mathematics.*

The quantic auxiliary to the multilinear operator

$$(\mu, \nu ; m, n)$$

is such that all its seminvariants are solutions of the partial differential equation

$$(\mu, \nu ; m, n) = 0 ;$$

and, conversely, every solution of this equation is a rational, but not necessarily integral, function of the seminvariants of the auxiliary quantic. This theorem is set forth as follows :—

Recalling the result (3) of Section IV., viz.,

$$(\mu, \nu ; m, n) A_{s,m'} = \left\{ (m'+m-1) \frac{\mu}{m} + (s-n) \nu \right\} A_{s-n, m'+m-1};$$

and observing that $a_{s+1} = A_{s+1,1}$, we find, by repeated operation of $(\mu, \nu ; m, n)$, the following set of relations :

$$(\mu, \nu ; m, n) a_{s+1} = \{\mu + (s-n+1)\nu\} A_{s-n+1, m},$$

$$(\mu, \nu ; m, n)^2 a_{s+1} = \{\mu + (s-n+1)\nu\} \times \left\{ (2m-1) \frac{\mu}{m} + (s-2n+1) \nu \right\} A_{s-2n+1, 2m-1},$$

$$(\mu, \nu ; m, n)^3 a_{s+1} = \{\mu + (s-n+1)\nu\} \left\{ (2m-1) \frac{\mu}{m} + (s-2n+1) \nu \right\} \times \left\{ (3m-2) \frac{\mu}{m} + (s-3n+1) \nu \right\} A_{s-3n+1, 3m-2},$$

... ...

where generally the j^{th} factor in the value of $(\mu, \nu ; m, n)^j a_{s+1}$ is

$$(jm-j+1) \frac{\mu}{m} + (s-jn+1) \nu.$$

Supposing s of the form $tn-1$, we find ultimately

$$(\mu, \nu ; m, n)^t a_{tn} = A_{0, tm-t+1} \cdot \prod_{\kappa=1}^{\kappa=t} \left\{ (\kappa m - \kappa + 1) \frac{\mu}{m} + (t-\kappa) n \nu \right\}.$$

Consider now the quantic of the t^{th} order,

$$\frac{1}{t!} (\mu, \nu ; m, n)^t a_{tn} X^t - \frac{1}{(t-1)!} (\mu, \nu ; m, n)^{t-1} a_{tn} X^{t-1} Y + \ldots + (-)^{t+1} \frac{1}{1!} (\mu, \nu ; m, n) a_{tn} X Y^{t-1} + (-)^t a_{tn} Y^t,$$

or, in a symbolic form,

$$(-Y)^t \exp\left\{-\frac{(\mu, \nu;\ m, n)\, X}{Y}\right\} a_{nt};$$

it is convenient also to denote it by

$$\frac{1}{t!}(B_0, B_1, B_2, \ldots B_t)(X, Y)^t,$$

and then $B_s = (-)^s s!\,(\mu, \nu;\ m, n)^{t-s} a_{tn}$.

We now have $(\mu, \nu;\ m, n) f(B_0, B_1, B_2, \ldots B_t)$

$$= \frac{df}{dB_0}(\mu, \nu;\ m, n) B_0 + \frac{df}{dB_1}(\mu, \nu;\ m, n) B_1 + \ldots + \frac{df}{dB_t}(\mu, \nu;\ m, n) B_t;$$

whence, since $(\mu, \nu;\ m, n) B_s = -sB_{s-1}$,

we see

$$(\mu, \nu;\ m, n) f(B_0, B_1, \ldots B_t) = -(B_0\partial_{B_1} + 2B_1\partial_{B_2} + 3B_2\partial_{B_3} + \ldots + tB_{t-1}\partial_{B_t}) f;$$

and, observing that the right-hand side is the seminvariant operator of the binomial quantic *quâ* the system of elements $B_0, B_1, \ldots B_t$, we are enabled to enunciate the following theorem:

"Every seminvariant of the binary quantic

$$(B_0, B_1, B_2, \ldots B_t)(X, Y)^t,$$

where $B_s = (-)^s s!\,(\mu, \nu;\ m, n)^{t-s} a_{tn}$,

is a solution of the multilinear partial differential equation

$$(\mu, \nu;\ m, n) = 0."$$

The algebraic value of B_s is, by what has gone before,

$$B_s = (-)^s s!\, A_{sn,\,(t-s)(m-1)+1} \prod_{j=1}^{j=t-s} \left\{(jm - j + 1)\frac{\mu}{m} + (t-j)\, n\nu\right\}.$$

A series of associated quantics is thus formed by giving t all positive integral values in succession; covariants of any of these quantics may be formed in any of the ways customary in the theory of invariants, and all the seminvariants which these covariants originate are solutions of the multilinear partial differential equation

$$(\mu, \nu;\ m, n) = 0;$$

conversely, every solution of this equation is a rational function of the seminvariants of the associated quantics.

In the theory of invariants, the series is the set of binomial quantics; whilst, in the theory of pure reciprocants, the series is derived by expressing $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ in succession, in terms of the differential coefficients $\frac{dx}{dy}, \frac{d^2x}{dy^2}, \frac{d^3x}{dy^3}, \ldots$ according to Burmann's law for the change of the independent variable. This series is set forth by Leudesdorf (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 329).

XIII. *The particular case of* $n = 1$.

When $n = 1$, the auxiliary may be written

$$\frac{\mu}{m} a^{t(m-1)+1} X^t - t.1!\, a^{(t-1)(m-1)} b X^{t-1} Y$$

$$+ \frac{t(t-1)}{2!} 2! \frac{a^{(t-2)(m-1)} c + \frac{(t-2)(m-1)}{2!} a^{(t-2)(m-1)-1} b^2}{\frac{(t-1)(m-1)+1}{m}\mu + \nu} X^{t-2} Y^2$$

$$- \frac{t(t-1)(t-2)}{3!} 3!$$

$$\times \frac{\left\{ \begin{array}{l} a^{(t-3)(m-1)} d + (t-3)(m-1)\, a^{(t-3)(m-1)-1} bc \\ + \frac{\{(t-3)(m-1)\}\{(t-3)(m-1)-1\}}{3!} a^{(t-3)(m-1)-2} b^3 \end{array} \right\}}{\left\{\frac{(t-1)(m-1)+1}{m}\mu + \nu\right\}\left\{\frac{(t-2)(m-1)+1}{m}\mu + 2\nu\right\}} X^{t-3} Y^3$$

$$+ \ldots\ldots .$$

Of this quantic we may form protomorphs, and, rejecting therefrom factors, which are numerical or powers of a, we may form a series of forms of degree-weight of which the earliest two are found to be

$$2\mu ac - m(\mu+\nu) b^2,$$

$$6\mu^2 a^2 d - 6m\mu(\mu+2\nu) abc + m(\mu+2\nu)\{m(\mu+2\nu)+\mu\} b^3,$$

which for $m = \mu = \nu = 1$ become

$$2(ac - b^2),$$

$$6(a^2 d - 3abc + 2b^3)$$

the ordinary seminvariants; and for $m = 2$, $\mu = 4$, $\nu = 1$,

$$2\,(4ac - 5b^2),$$

$$96\,(a^2d - 3abc + 2b^3),$$

the pure reciprocant protomorphs; whilst, for $m = \mu = 1$, $\nu = 0$, we find

$$2ac - b^2,$$

$$2\,(3a^2d - 3abc + b^3),$$

the well-known non-unitary symmetric functions.

On the Theory of Screws in Elliptic Space. Fourth Note.

By A. Buchheim, M.A.

[*Read Nov. 11th*, 1886.]

In this fourth, and, I hope, last Note on Screws in Elliptic Space, I extend some of the results of my Third Note to space of n dimensions. I have tried to vary the methods used as far as possible, in order that the extension should not consist merely in the addition of subscript indices and signs of summation, but should subserve an object that I have had in view all along, namely, the illustration and extension of Grassmann's methods. I have only succeeded in extending the results contained in the first part of my former Note. It does not seem possible to extend the application of Biquaternions to the representation of motions, for the simple reason that, as soon as we get beyond quaternions it becomes impossible to write the product of two quantities of the same kind as a new quantity of the same kind, unless we use one of Peirce's Algebras; but I do not think we should gain anything by doing so.

I.

In what follows I denote the complement of any quantity a by Ka, so that, if a is a product of units and the product of all the units is unity, we have $aKa = 1$. I denote by ω the matrix of any equation of the absolute, so that, if a' is the conjugate of a with respect to the absolute, and if ω denotes the matrix of the equation of the absolute

corresponding to the class of quantity to which a belongs (point equation if a is a point, line equation if a is a line, and so on), we have

$$a' = K\omega a.$$

I use Sab to denote $aK\omega b = bK\omega a$, and T^2a for Saa, and I do not use any special symbol for aKb when the matrix of the absolute is not the unit matrix.

If ϕ is any matrix there is a certain matrix ϕ' corresponding to it, and defined by the equation $Sx\phi' y = Sy\phi x$, x, y being any two quantities of the same kind. This matrix I call the conjugate of ϕ. If we suppose ϕ written in the form of a square array, there is another matrix corresponding to it, which is generally called its conjugate. I shall call this the reflection of ϕ, and denote it by ϕ''; the equation defining ϕ'' is

$$xK\phi'' y = yK\phi x.$$

It is easy to show that we have $\phi' = \omega^{-1}\phi''\omega$, for we have

$$\begin{aligned} Sx\phi' y &= Sy\phi x \\ &= yK\omega\phi x \\ &= xK\phi''\omega y^* \\ &= xK\omega \,.\, \omega^{-1}\phi''\omega y \\ &= Sx\omega^{-1}\phi''\omega y, \end{aligned}$$

and therefore $\phi' = \omega^{-1}\phi''\omega$.

If $\phi'' = \phi$, ϕ is symmetric; and if $\phi'' = -\phi$, ϕ is skew-symmetric; if $\phi' = \phi$ I call ϕ self-conjugate, and if $\phi' = -\phi$ I call ϕ skew-conjugate.

If ϕ is symmetric or skew-symmetric, $\omega^{-1}\phi$ is self-conjugate or skew-conjugate.

For

$$\begin{aligned} (\omega^{-1}\phi)' &= \omega^{-1}(\omega^{-1}\phi)''\omega \\ &= \omega^{-1}\phi''\omega^{-1}\omega \\ &= \omega^{-1}\phi'', \end{aligned}$$

and therefore, if $\phi'' = \pm\phi$, $(\omega^{-1}\phi)' = \pm\omega^{-1}\phi$.

It is easy to see that we have $(ab)' = b'a'$.

For

$$\begin{aligned} (ab)' &= \omega^{-1}(ab)''\omega \\ &= \omega^{-1}b''a''\omega \\ &= \omega^{-1}b''\omega\,\omega^{-1}a''\omega \\ &= b'a'. \end{aligned}$$

* Since ω is symmetric.

II.

Let ϕ be any matrix, and let ω be the matrix of the point equation of the absolute, then the equation of the absolute is $xK\omega x = 0$. If we transform space by means of the operator ϕ, the point ϕx is on the absolute if $\phi x K \omega \phi x = 0$, that is, if $xK\phi''\omega\phi x = 0$. Therefore, if the matrix ϕ is to represent a motion, that is, if it is not to alter the absolute, we must have $\phi''\omega\phi = \lambda\omega$, where λ is a scalar which may always be supposed equal to unity. Now, if $\phi''\omega\phi = \omega$, we have $\omega^{-1}\phi''\omega\phi = 1$, that is, $\phi'\phi = 1$. It follows at once from this that, if

$$\psi = \frac{1-\phi}{1+\phi},$$

we have $$\psi + \psi' = 0,$$

so that ψ is skew-conjugate. We see, therefore, that the matrix defining a motion can be written in the form $\frac{1-\psi}{1+\psi}$, where ψ is a skew-conjugate matrix. Moreover, we saw in (2) that, if χ is a skew-symmetric matrix, $\omega^{-1}\chi$ is skew-conjugate, and conversely; we can therefore write $\omega^{-1}\chi$ for ψ, and we get Prof. Cayley's expression for the automorphics of a quadric ω in the form

$$\frac{1-\omega^{-1}\chi}{1+\omega^{-1}\chi},$$

where χ is a skew-symmetric matrix.

III.

In an n-tuple space (n-point, space of $n-1$ dimensions) a screw is defined as a form of order $n-2$. If a is any screw, I write

$$a = \Sigma a_{ik} K e_i e_k,$$

where the summation extends to the different pairs that can be formed from the n indices 1, 2, ... n, and where we need not be careful as to the order of e_i, e_k, if we assume $a_{ik} = -a_{ki}$. If we multiply a point by a screw, we get a form of order $n-1$ whose n coefficients are derived from those of the point by a skew-symmetric matrix. For, let $x = \Sigma v_i e_i$ be any point; then we see that in the product xa the coefficient a_{ik} will occur twice, namely, once in the term $a_{ik}x_i e_i K e_i e_k$, and again in the term $a_{ik}x_k e_k K e_i e_k$. Now $e_i[e_k K e_i e_k] = 1$, and $e_k[e_i K e_i e_k] = -1$. We see, therefore, that the two terms in which a_{ik} occurs are $-a_{ik}x_i K e_k$ and $a_{ik}x_k K e_i$ respectively, and that, if we write $xa = \Sigma \xi_i K e_i$, the coefficient of x_k in the value of ξ_i is a_{ik}, while

the coefficient of x_i in the value of ξ_k is $-a_{ik} = a_{ki}$; it follows that we have $(\xi_1 \dots \xi_n) = (A \times x_1 \dots x_n)$, where $A = \|a_{ik}\|$, and is skew-symmetric.

I call A the matrix appertaining to the screw a, and a the screw appertaining to the matrix A.

The analytical connection between a screw and a motion can now be stated as follows:—Starting with the screw a, we have a skew-symmetric matrix A appertaining to it, and the matrix defining the motion is

$$\phi = \frac{1-\omega^{-1}A}{1+\omega^{-1}A}.$$

If we start with a motion defined by a matrix ϕ, we have a skew-symmetric matrix

$$\omega\frac{1-\phi}{1+\phi},$$

and the screw appertaining to this matrix is the screw defining the motion.

IV.

We have now to reduce the motion to a canonical form. In my former Note, I started with a skew-conjugate matrix and reduced this to its canonical form. In the present Note, I start with the (orthogonal) matrix ϕ. Let $\lambda_1 \dots \lambda_n$ be its latent roots, supposed unequal, and let $e_1 \dots e_n$ be the latent points appertaining to them, so that

$$\begin{aligned} \phi e_i &= \lambda_i e_i, \\ \phi' e_i &= \phi^{-1} e_i \\ &= \lambda_i^{-1} e_i. \end{aligned}$$

We see that, unless $\lambda_i = \pm 1$,

$$T^2 e_i = 0.$$

For we must have

$$\begin{aligned} \lambda_i T^2 e_i &= S e_i \phi e_i \\ &= S e_i \phi' e_i \\ &= \lambda_i^{-1} T^2 e_i. \end{aligned}$$

Moreover,

$$\begin{aligned} S e_i \phi e_k &= \lambda_k S e_i e_k \\ &= S e_k \phi' e_i \\ &= \lambda_i^{-1} S e_i e_k, \end{aligned}$$

and therefore, unless

$$\lambda_i \lambda_k = 1,$$

$$S e_i e_k = 0.$$

It follows that all points appertaining to latent roots not equal to unity are on the absolute, and that any two points appertaining to latent roots whose product is not unity are conjugate with respect to the absolute, so that if they are both on it the line joining them is a generator. Now it is easy to see, by considering the form of ω, that unless the absolute is a degenerate quadric it is impossible to take n asyzygetic points all on the absolute, such that the lines joining any two of them are generators,* and that if one of a set of n asyzygetic points is on the absolute there must be at least one other point of the set which is not on the same generator.

It follows from this that, in general, if $n = 2\mu$, the latent points group themselves in pairs answering to latent roots that are reciprocals, and that if $n = 2\mu+1$ we have a latent root unity,† and the points appertaining to the remaining latent roots group themselves in pairs.

If $n = 2\mu$, I write, for the canonical form of ϕ,

$$\phi e_i = \lambda_i e_i ,$$

$$\phi e_{\mu+i} = \lambda_i^{-1} e_{\mu+i} \ (i = 1 \dots \mu).$$

If $n = 2\mu+1$, I write

$$\phi e_\infty = e_\infty ,$$

$$\phi e_i = \lambda_i e_i ,$$

$$\phi e_{\mu+i} = \lambda_i^{-1} e_{\mu+i} \ (i = 1 \dots \mu),$$

and we have $Se_i e_k = 0$, unless $k = \mu+i$, and $T^2 e_i = 0$ unless $i = \infty$. We have of course $e_i = e_k$ if $i \equiv k \pmod{2\mu}$.

V.

I shall now transform the equations of the absolute, taking $e_\infty \dots e_{2\mu}$ as points of reference.‡ I shall consider the case in which n is odd, since we can always pass from $n = 2\mu+1$ to $n = 2\mu$, by omitting the terms involving e_∞.

We have at once, if x is a point,

$$T^2 x = T^2 (x_\infty e_\infty + x_1 e_1 + \dots + x_{2\mu} e_{2\mu})$$

$$= x_\infty^2 T^2 e_\infty^2 + 2\Sigma x_i x_k Se_i e_k$$

$$= x_\infty^2 T^2 e_\infty + 2\Sigma x_i x_{\mu+i} Se_i e_{\mu+i},$$

* For $T^2(x_1 e_1 + \dots + x_n e_n) = \Sigma x_i^2 T^2 e_i + 2\Sigma x_i x_k Se_i e_k \equiv 0$ if $T^2 e_i$, $Se_i e_k$ vanish for all values of i, k, that is, all points of space would be on the absolute.

† If a latent root is -1, we can take $-\omega$ instead of ω. If $+1$ and -1 are latent roots, the motion is a particular case, and the screw belonging to it is indeterminate.

‡ These transformations are effected in an unnecessarily clumsy way in Section III. of my third Note.

since all the other terms disappear in virtue of the relations connecting the e's. I write (∞) for $T^2 e_\infty$, and (i) for $Se_i e_{\mu+i}$, and we therefore have

$$T^2 x = x_\infty^2 (\infty) + 2\Sigma (i)\, x_i x_{\mu+i}.$$

To get the remaining equations, I use the theorem

$$S\left[a_1 a_2 \dots a_k\right]\left[\beta_1 \beta_2 \dots \beta_k\right] = \text{Det.} \mid Sa_i\beta_k \mid .*$$

We get
$$S\left[e_i e_k\right]\left[e_l e_m\right] = \begin{vmatrix} Se_i e_l, & Se_i e_m \\ Se_k e_l, & Se_k e_m \end{vmatrix}$$
$$= 0,$$

unless
$$\left.\begin{aligned} l &= \mu + i \\ m &= \mu + k \end{aligned}\right\},$$

or
$$\left.\begin{aligned} l &= i = \infty \\ m &= \mu + k \end{aligned}\right\},$$

and is $= (i)(k)$, or $(\infty)(k)$, in these two cases.

$$T^2\left[e_i e_k\right] = \begin{vmatrix} T^2 e_i, & Se_i e_k \\ Se_i e_k, & T^2 e_k \end{vmatrix}$$
$$= 0,$$

unless
$$k = \mu + i,$$

and
$$= -(i)^2,$$

if this condition is satisfied.

I write $[i, k, l \dots]$ for the coefficient of $[e_i e_k e_l \dots]$ in any quantity. We see that, if x is a line,

$$T^2 x = -\sum_1^\mu (i)^2 \left[i, \mu+i\right]^2 + 2\sum_{i,k=1\dots\mu} (i)(k)\left[i,k\right]\left[\mu+i, \mu+k\right]$$
$$+ 2(\infty)\sum_{i,k=1\dots\mu} (i)\left[\infty, i\right]\left[\infty, \mu+i\right].$$

In general, we get

$$S\left[e_i e_k e_l \dots e_m\right]\left[e_{\mu+i} e_{\mu+k} e_{\mu+l} \dots e_{\mu+m}\right] = (i)(k)(l)\dots(m),$$
$$S\left[e_\infty e_k e_l \dots e_m\right]\left[e_\infty e_{\mu+k} e_{\mu+l} \dots e_{\mu+m}\right] = (\infty)(k)(l)\dots(m),$$
$$T^2\left[e_i e_{\mu+i} \dots e_m e_{\mu+m}\right] = (-)^{\frac{1}{2}r(r-1)} (i)^2 (k)^2 \dots (m)^2,$$
$$T^2\left[e_\infty e_i e_{\mu+i} \dots e_m e_{\mu+m}\right] = (-)^{\frac{1}{2}(r-1)(r-2)} (\infty)^2 (i)^2 \dots (m)^2,$$

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* *Grassmann*, Ausdehnungslehre (1862), § 175. Grassmann only proves this for the case $\omega = 1$, but the extension is obvious.

where r is the order of the form, and these are the only terms that do not vanish. We have therefore, for forms of order $2r$,

$$T^2x = (-)^r \Sigma (i)^2 (k)^2 \ldots\ldots [i, \mu+i, k, \mu+k \ldots\ldots]^2$$
$$+2\Sigma (i)(k) \ldots\ldots [i, k \ldots\ldots][\mu+i, \mu+k \ldots\ldots]$$
$$+2(\infty)\Sigma (i)(k) \ldots\ldots [\infty, i, k \ldots\ldots][\infty, \mu+i, \mu+k \ldots\ldots],$$

and for forms of order $2r+1$

$$T^2x = (-)^r \Sigma (\infty)^2 (i)^2 (k)^2 \ldots\ldots [\infty, i, \mu+i, k, \mu+k \ldots\ldots]^2$$
$$+2\Sigma (i)(k) \ldots\ldots [i, k \ldots\ldots][\mu+i, \mu+k \ldots\ldots]$$
$$+2(\infty)\Sigma (i)(k) \ldots\ldots [\infty, i, k \ldots\ldots][\infty, \mu+i, \mu+k \ldots\ldots],$$

where the indices take the values 1, 2, ... μ.

VI.

I now return to the motion. Let x be the old position of any point, x' its new position; then we have

$$x = \Sigma (x_i e_i + x_{\mu+i} e_{\mu+i}) + x_\infty e_\infty,$$
$$x' = \Sigma (\lambda_i x_i e_i + \lambda_i^{-1} x_{\mu+i} e_{\mu+i}) + x_\infty e_\infty,$$

and therefore

$$Sxx' = \Sigma (i) x_i x_{\mu+i} (\lambda_i + \lambda_i^{-1}) + (\infty) x_\infty^2,$$
$$Tx' = Tx,$$

and therefore

$$\cos (xx') = \Sigma \frac{(i) x_i x_{\mu+i}}{T^2x} (\lambda_i + \lambda_i^{-1}) + \frac{(\infty) x_\infty^2}{T^2x}$$
$$= \Sigma \frac{2 (i) x_i x_{\mu+i}}{T^2x} \cos\theta_i + \frac{(\infty) x_\infty^2}{T^2x},$$

if $$\cos\theta_i = \frac{\lambda_i + \lambda_i^{-1}}{2}.$$

Now
$$T^2 [x e_i e_{\mu+i}] = \begin{vmatrix} T^2x, & Sxe_i, & Sxe_{\mu+i} \\ Sxe_i, & 0, & (i) \\ Sxe_{\mu+i}, & (i), & 0 \end{vmatrix}$$
$$= (i)^2 \{2 (i) x_i x_{\mu+i} - T^2x\},$$

and
$$T^2 [e_i e_{\mu+i}] = -(i)^2.$$

Therefore $$\sin^2(xe_i^{!}e_{\mu+i}) = \frac{T^2\left[xe_ie_{\mu+i}\right]}{T^2xT^2\left[e_ie_{\mu+i}\right]}$$

$$= \frac{T^2x - 2(i)x_ix_{\mu+i}}{T^2x},$$

and $$\cos^2(xe_ie_{\mu+i}) = \frac{2(i)x_ix_{\mu+i}}{T^2x}.$$

Moreover, $$Sxe_\infty = (\infty)x_\infty,$$

and therefore $$\cos^2(xe_\infty) = \frac{x_\infty^2}{T^2x}.$$

Remarking that, if a point is on $e_ie_{\mu+i}$ it obviously remains on this line and is moved through a distance θ_i, and that e_i remains fixed, we get the following theorem :—

Every motion of a rigid body in a space of $2\mu-1$ dimensions consists of translations along μ axes at right angles to each other; if θ_i is the distance through which a point on one of the axes is moved, we have $\theta_i = \cos^{-1}\frac{(\lambda_i+\lambda_i^{-1})}{2}$, λ_i being a latent root of the matrix defining the motion. The general expression for the distance through which any point is moved is $\cos^{-1}\Sigma\cos^2 D_i\cos\theta_i$, D_i being the distance of the point from the i^{th} axis. Every motion of a rigid body in a space of 2μ dimensions consists of a rotation about a fixed point, and translations along 2μ lines. A point moves through a distance given by the equation $\cos\theta = \Sigma\cos^2 D_i\cos\theta_i + \cos^2 D$, where D is its distance from the fixed point, D_i its distance from an axis points on which move through the distance $\theta_i = \cos^{-1}\frac{(\lambda_i+\lambda_i^{-1})}{2}$, λ_i being a latent root (not unity) of the matrix defining the motion.

VII.

I shall now show how the preceding investigations lead to a reduction of the screw defining the motion to a canonical form.

I write ν for the latent roots of the skew-conjugate matrix appertaining to the motion. Then, since the latent points of this skew-conjugate matrix ψ are the same as those of the orthogonal matrix ϕ, for

$$\phi = \frac{1-\psi}{1+\psi},$$

and since, A being the matrix of the screw a defining the motion, we

have $\psi = \omega^{-1}A$, we see that the points e_i satisfy the equations

$$\omega^{-1}Ae_i = \nu_i e_i,$$

$$\omega^{-1}Ae_{\mu+i} = -\nu_i e_{\mu+i},$$

$$\omega^{-1}Ae_\infty = 0,$$

or

$$Ae_i = \nu_i \omega e_i,$$

$$Ae_{\mu+i} = -\nu_i \omega e_{\mu+i},$$

$$Ae_\infty = 0;$$

and therefore, by what was proved in (2),

$$e_i a = \nu_i K\omega e_i = (i)\ \nu_i K e_{\mu+i},$$

$$e_{\mu+i}a = -\nu_i K\omega e_{\mu+i} = -\nu_i\ (i)\ K e_i,$$

$$e_\infty a = 0,$$

since it follows, from the form of ω referred to $e_\infty\ e_i \ldots\ldots e_{2\mu}$, that $\omega e_i = (i)\ e_{\mu+i}$. We see, then, that we must have

$$a = -\Sigma\nu_i\ (i)\ K\left[e_i e_{\mu+i}\right],^*$$

whether $n = 2\mu$ or $2\mu+1$, and we can, if we choose, omit the negative sign. This is the canonical form of the screw a. It is obvious that we might have started by reducing $\omega^{-1}A$ to a canonical form, thus determining the axes $e_i e_{\mu+i}$ of the screw; and that we should have found that the motion consisted of translations along the axes of the screw, accompanied in the case of a space of an even number of dimensions by a rotation about a fixed point; and we see (since the fixed point belongs to the screw as much as the axes) that for space of any number of dimensions a motion consists of a twist about a screw.

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* Since we can divide the points e_i by any scalars, we see that we can make all the (i) equal to unity. If we do this, the product of all the units, which is $\surd[(i)\ (k)\ldots\ldots]$, will be unity.

On the Rectification of Certain Curves. *By* R. A. Roberts, M.A.

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1. I purpose in this paper to give some results concerning the rectification of algebraic curves, both plane and twisted. Amongst these is the rectification of a series of twisted unicursal curves, whose arcs in particular cases are shown to afford a very general representation of elliptic and hyper-elliptic integrals. I arrive at several theorems also involving geometrical relations between the arcs of curves in cases in which these arcs are expressible by elliptic integrals.

2. Before proceeding, however, to the main subject of this paper, it may be of interest to give a brief account of what has been done in the subject of the rectification of curves, and to mention those points which have been of the most interest to geometers. J. Bernoulli and Euler appear to have been the first to consider the question of finding algebraic curves whose arcs should represent given integrals. The latter discovered certain curves whose arcs, measured from a fixed point, are equal to a single circular function (see "New Memoirs of St. Petersburg," t. xi., and Serret's "Cours de Calcul Différentiel et Intégral," t. ii., p. 252). These curves derived their interest from the fact that their arcs could be multiplied and divided algebraically, exactly in the same manner as those of the circle.

3. Legendre then, after having noticed that the arc of the lemniscate of Bernoulli was expressible by an elliptic integral of the first kind with a modulus equal to $\sqrt{(\frac{1}{2})}$ (a result first given by Fagnani), appears to have attempted to find a curve whose arc could be expressed by a similar integral with an arbitrary modulus, as by the theory of these functions the arc of such a curve could be multiplied and divided algebraically. In this he was unsuccessful, his nearest approach to a solution being the curve enveloped by a perpendicular to the central radius vector of an ellipse at its extremity. This curve is of the fourth class and sixth degree. The actual determination of its Cartesian equation baffled Legendre, and was accomplished afterwards by Tortolini (*Crelle*, t. 32, p. 90); but the result can be easily obtained by taking the discriminant of a binary quartic. The solu-

tion is, however, evidently imperfect, as the arc of the curve is an elliptic integral together with an algebraic quantity, and therefore cannot be divided or multiplied algebraically.

Not long afterwards Abel gave some interesting investigations, in which he showed how to sub-divide algebraically the arc of the lemniscate, that is, an integral of the kind mentioned above. Amongst his results was included the resolution of the semi-perimeter into seventeen parts (*Crelle*, t. IV., p. 131; see also Libri, *Crelle*, t. X., p. 167).

4. At the time, then, that Serret took up the problem, the lemniscate was the only curve known whose arc exactly represented an elliptic integral of the first kind. In his first paper on the subject (Liouville's *Journal*, t. VIII., p. 145), Serret showed that the arc of the ellipse of Cassini was expressible by the sum or difference of two elliptic integrals of the first kind with complementary moduli, and hence that an arc of the curve with variable extremities represented exactly an elliptic integral. This is the simplest representation we have in a plane, but, since the extremities of the arc which is equal to the single integral are variable, we cannot divide or multiply algebraically any assigned arc of the curve, and therefore the solution is incomplete. Shortly afterwards my father, the Rev. W. Roberts, gave a spherical curve of the fourth order whose arc, measured from a fixed point, is equal to an elliptic integral of the first kind with an arbitrary modulus (Liouville's *Journal*, t. VIII., p. 263). Since then, it may be observed that several other twisted curves have been found whose arcs can be expressed in the same manner (see Hermite's *Cours d'Analyse de l'Ecole Polytechnique*, p. 421). The plane curves, however, remained still to be found, and it was to this problem that Serret especially directed his attention (see Cayley's "Elliptic Functions," Chap. 15). He discovered a series of curves which are algebraic provided a certain number n is commensurable. The arc, then, is an elliptic integral of the first kind whose modulus is a function of n. Thus, for each curve corresponding to an assigned value of n, the arc represents an elliptic integral with a numerical modulus. This solution of Serret's is in all probability the best to be obtained, but still it evidently falls short of what is desirable.

5. Since then, it may be noticed, my father gave two very simple plane curves whose arcs are expressible by elliptic integrals of the first kind with certain numerical moduli (see *Journal de Mathematiques*, t. XII., p. 445). We see thus that geometers directed their special attention to finding curves whose arcs can be multiplied or

divided algebraically, that is, are expressible by a circular function or an elliptic integral of the first kind. The case of the logarithm seems to have been overlooked; but there does not appear to be much difficulty in finding curves whose arcs are expressible as a single logarithmic function.

6. With reference to curves whose arcs represent other integrals, we may notice that, as the arc of the plane ellipse represents the elliptic integral of the second kind, so the arc of the spherical conic represents the elliptic integral of the third kind with a circular parameter (Gudermann, *Crelle*, t. XIV., p. 169). One of the most remarkable results in the rectification of curves was the discovery by an Italian geometer, Genocchi, that the arc of the Cartesian oval was equal to the sum of the arcs of three ellipses (see Tortolini's *Annali*, 1864, p. 97). Since then, Dr. Casey has shown by geometrical considerations that the arc of the general bicircular quartic can be resolved into the sum of four quantities, each of which can be expressed by elliptic integrals. (See his Memoir in the *Philosophical Transactions of the Royal Society*, Vol. 167, Pt. 2, and Professor Cayley's addition thereto.) This reduction, however, in a certain case, is not effected by real transformations; so that the arc then depends upon elliptic integrals with imaginary moduli, and has to be expressed in a real manner by hyper-elliptic integrals of the first kind; a result which, I believe, has not been noticed before, and which I proceed to demonstrate here.

7. Let $$F \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be the equation of a focal conic, and

$$J \equiv (x-\alpha)^2 + (y-\beta)^2 - k^2 = 0$$

that of the corresponding Jacobian circle (see Casey's Memoir on Bicircular Quartics); then, if r, ϑ are the polar coordinates of a point on the quartic referred to the centre of J, we have, from the theory of the curve, $$r = p + \sqrt{(p^2-k^2)}, \quad \vartheta = \omega,$$

where p is the perpendicular from α, β on a tangent to F, and ω is the angular coordinate of that tangent. Hence

$$ds = \sqrt{(dr^2 + r^2 d\vartheta^2)} = \left\{1 + \frac{p}{\sqrt{(p^2-k^2)}}\right\} \left\{p^2 + \left(\frac{dp}{d\omega}\right)^2 - k^2\right\}^{\frac{1}{2}} d\omega;$$

but, if $a\cos\vartheta$, $b\sin\vartheta$ are the coordinates of the point on the conic,

we have $$p = \frac{ba\cos\vartheta + a\beta\sin\vartheta - ab}{\sqrt{(a^2\sin^2\vartheta + b^2\cos^2\vartheta)}},$$

$$d\omega = \frac{ab\,d\vartheta}{a^2\sin^2\vartheta + b^2\cos^2\vartheta},$$

and $$p^2 + \left(\frac{dp}{d\omega}\right)^2 = (\alpha + a\cos\vartheta)^2 + (\beta + b\sin\vartheta)^2.$$

We thus obtain for ds an expression of the form

$$\frac{P\,d\vartheta}{\sqrt{U}} + \frac{Q\,d\vartheta}{\sqrt{UV}},$$

where P, Q are rational functions of $\cos\vartheta$, $\sin\vartheta$, and

$$U \equiv (\alpha + a\cos\vartheta)^2 + (\beta + b\sin\vartheta)^2 - k^2,$$

$$V \equiv (ba\cos\vartheta + a\beta\sin\vartheta - ab)^2 - k^2\,(a^2\sin^2\vartheta + b^2\cos^2\vartheta).$$

Now, the first part of this value of s, namely,

$$\int \frac{P\,d\vartheta}{\sqrt{U}},$$

evidently depends upon elliptic integrals; we therefore confine our attention to the latter part. Putting $\tan\frac{1}{2}\vartheta = \mu$, this integral becomes

$$I = \int \frac{R\,d\mu}{\sqrt{UV}},$$

where R is a rational function of μ, and U, V are two binary quartics in μ. Now, it is easy to see the geometrical meaning of U and V; U, in fact, determines the parameters of the rays joining a point P on F to the points of intersection of F with J, and V determines the rays joining P to the points on H at which the tangents touch J. But, referring F and J to their common self-conjugate triangle, that is, writing

$$F = x^2 + y^2 - z^2, \quad J = ax^2 + by^2 - cz^2,$$

and putting then $x = \lambda^2 - \mu^2$, $y = 2\lambda\mu$, $z = \lambda^2 + \mu^2$ for a point on F, we see that U is of the form

$$\lambda^4 + \mu^4 + 6m\lambda^2\mu^2,$$

and that V is of the same form, that is, can be written $U + \lambda H$, where H is the Hessian of U; for it arises from the intersection of F with the conic

$$bcx^2 + cay^2 + abz^2 = 0.$$

Now, if the four intersections of F and J are all real or all imaginary, we may write

$$U \equiv \mu^4+6m\mu^2+1, \qquad V \equiv \mu^4+6n\mu^2+1.$$

Hence, putting $$R = A+\mu B,$$

where A, B are functions of μ^2, we have

$$I = \int \frac{A+B\sqrt{y}}{\sqrt{(y^2+6my+1)(y^2+6ny+1)}},$$

where $\mu^2 = y$. Now, part of this integral depends upon elliptic integrals, and the remaining part upon

$$\int \frac{yB\,dy}{\sqrt{\{y\,(y^2+6my+1)(y^2+6ny+1)\}}}.$$

But, putting $$y+\frac{1}{y} = z,$$

this latter expression is found to depend upon two elliptic integrals with different moduli. That is, if the four points of intersection of F and J are either all real or all imaginary, or, which is the same thing, if the four real foci of the curve are concyclic, the arc of the quartic can be broken up into the sum of four expressions, each of which can be expressed by elliptic integrals. This is Dr. Casey's result. If, however, the real foci of the curve are not concyclic, two points of intersection of F and J will be real and two imaginary, and U, V may be reduced to the forms

$$U \equiv \mu^4+6m\mu^2-1, \qquad V \equiv \mu^4+6n\mu^2-1,$$

respectively. Hence, in this case, we have

$$I = \int \frac{A+B\sqrt{y}}{\sqrt{\{(y^2+6my-1)(y^2+6ny-1)\}}},$$

where $\mu^2 = y$, and, as before, the first part depends upon elliptic integrals; but the second part, namely,

$$\int \frac{yB\,dy}{\sqrt{\{y\,(y^2+6my-1)(y^2+6ny-1)\}}},$$

cannot be reduced in a real manner any further. This result, we see depends upon the fact that the vanishing of the skew invariant E of a sextic X is not a sufficient condition that

$$\int \frac{R\,dx}{\sqrt{X}}$$

should break up into elliptic integrals with real moduli. It is also necessary that the double points of the involution determined by the three quadratic factors of X should be real.

8. As I have remarked already, geometers do not appear to have noticed those curves whose arcs represent a pure logarithmic function. I shall proceed here to obtain a few of such curves. First, I shall show that there are some simple curves of the fourth order whose arcs can be expressed in this manner.

Let us consider the inverse of the parabola from an arbitrary point, then the arc of this curve is equal to

$$\int \frac{ds}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where ds is the element of an arc of the parabola, and r is the distance of the element from the fixed point. Now, r^2 can be expressed rationally in terms of a parameter; hence we see that (1) depends upon logarithms and circular functions. Let us consider now the case in which the centre of inversion lies on the axis of the parabola, and see whether (1) can be expressed as a pure logarithm. Taking, then, $m\mu^2$, $2m\mu$ as the coordinates of a point on the parabola, and a, 0 as those of the centre of inversion, we have

$$s = \int \frac{2m\sqrt{(1+\mu^2)}\, d\mu}{(a+m\mu^2)^2+4m^2\mu^2}.$$

Hence, if a is greater than $3m$, we obtain

$$s = \frac{1}{2\sqrt{(m^2+ma)}} \left\{ p \log \left(\frac{1+p \sin \omega}{1-p \sin \omega} \right) - q \log \left(\frac{1+q \sin \omega}{1-q \sin \omega} \right) \right\} \ldots(2),$$

from $\mu = 0$ to $\mu = \tan \omega$, where

$$p^2 = \frac{a+m+2\sqrt{(m^2+ma)}}{a+2m+2\sqrt{(m^2+ma)}},$$

$$q^2 = \frac{a+m-2\sqrt{(m^2+ma)}}{a+2m-2\sqrt{(m^2+ma)}}.$$

In order, then, that the arc of this curve should represent the logarithm of an algebraic function of the variable, we must take the ratio of p to q commensurable, that is, if $a+m = m\lambda^2$, we must have

$$\frac{(\lambda+2)}{(\lambda-2)} \frac{(1-\lambda)^2}{(1+\lambda)^2} = \frac{s^2}{t^2},$$

where s and t are numbers. This gives a cubic equation to determine λ or α, when s, t are assigned, and the arc of the curve from (2) is then proportional to

$$\log\left\{\left(\frac{1+p\sin\omega}{1-p\sin\omega}\right)^{s}\left(\frac{1-q\sin\omega}{1+q\sin\omega}\right)^{t}\right\}.$$

The simplest case is obtained by taking $t=0$, when we get $a=3m$ and

$$s=\frac{1}{3m\sqrt{2}}\log\left\{\frac{3+2\sqrt{2}\sin\omega}{3-2\sqrt{2}\sin\omega}\right\}.$$

As an example, then, of the algebraic subdivision of the arcs of this curve, we may notice that the semiperimeter is bisected at the point corresponding to $\sin\omega=\frac{3}{4}$.

9. I now mention a system of curves whose arcs in each case represent either a logarithm or a circular function. Consider the hypocycloid whose equation in p, ω coordinates is $p=b\cos n\omega$ (see Salmon's "Curves," Art. 311). Then, for the arc s of the inverse of the given curve with respect to the origin, we have

$$s=\int\frac{\left(p+\frac{d^2p}{d\omega^2}\right)d\omega}{p^2+\left(\frac{dp}{d\omega}\right)^2}=\int\frac{(1-n^2)\cos n\omega\,d\omega}{b\,(\cos^2 n\omega+n^2\sin^2 n\omega)}\quad\ldots\ldots\ldots\ldots(3),$$

in the case we are considering. Hence, putting $\sin n\omega=z$, we get

$$s=\frac{1-n^2}{nb}\int\frac{dz}{1-(1-n^2)\,z^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Thus, if n be less than unity, we have a series of curves whose arcs represent a pure logarithm. For instance, for the envelope of the system of circles

$$b\,(x\cos\omega+y\sin\omega)-\cos\tfrac{1}{2}\omega\,(x^2+y^2)=0,$$

we get

$$s=\tfrac{1}{2}b\sqrt{3}\log\left(\frac{1+\frac{1}{2}\sqrt{3}\sin\frac{1}{2}\omega}{1-\frac{1}{2}\sqrt{3}\sin\frac{1}{2}\omega}\right).$$

Again, taking $n=\frac{1}{3}$, we have

$$s=2b\sqrt{2}\log\left(\frac{1+\frac{2}{3}\sqrt{2}\sin\frac{1}{3}\omega}{1-\frac{2}{3}\sqrt{2}\sin\frac{1}{3}\omega}\right).$$

In the latter case, the curve is an inverse of the cardioid, and therefore, it is easy to see, the inverse of a parabola; in fact, it is the

curve already considered above. By taking n greater than unity, we get curves whose arcs represent a circular function. Thus, for example, putting $n = 3$, the arc of the curve

$$b^4 + 8b^3 r \cos 3\vartheta + 18b^2 r^2 = 27r^4$$

(Salmon's "Curves," Art. 310, Ex. 5) is a circular function.

10. I now mention some methods of generating curves whose arcs can be expressed linearly in terms of arcs of a given curve, or of several given curves. This is the idea of John Bernoulli (see on "Curva Reptoria"). Consider several points $x_1 y_1$, $x_2 y_2$, &c., on a curve, or on different curves at which the tangents are parallel; then, if x_1, y_1 are the coordinates of the centre of mean position of these points for the system of multiples l, m, &c., we have

$$x = lx_1 + mx_2 + \&c., \quad y = ly_1 + my_2 + \&c.,$$

where

$$\Sigma l = 1.$$

Hence, since

$$\frac{dx_1}{ds_1} = \frac{dx_2}{ds_2} = \&c. = \cos\phi, \text{ say,}$$

$$\frac{dy_1}{ds_1} = \frac{dy_2}{ds_2} = \&c. = \sin\phi,$$

we have

$$dx = (l\,ds_1 + m\,ds_2 + \&c.)\cos\phi,$$

$$dy = (l\,ds_1 + m\,ds_2 + \&c.)\sin\phi;$$

from which we see that the tangent to the locus of xy is parallel to the other tangents, and that its arc is equal Σls. Hence, by taking l, m, &c. as numbers, we see that, if the arc of a curve be equal to a logarithm, a circular function, or an elliptic integral of the first kind, we can generate from this curve an infinite number of other curves whose arcs can be expressed in the same manner; for, by the properties of these functions, the several expressions for the different arcs can be compounded into a single expression which is a function of the same kind as is involved in each of the arcs.

As an example of this result, we may consider the locus of the middle point of the chord of contact of two parallel tangents to the lemniscate of Bernoulli. Putting $x + iy = u$, $x - iy = v$, in the equation

$$(x^2 + y^2)^2 = 2(x^2 - y^2),$$

we get

$$u^2 v^2 = u^2 + v^2.$$

Now, taking

$$u^2 = 1 + t^2, \quad v^2 = 1 + t^{-2} \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

we find that t^3 has the same value for parallel tangents; hence, for the two points on the curve, we may take $\vartheta\mu$, $\vartheta^2\mu$ as the parameters, where ϑ is a cube root of unity. Thus for the locus we have, from (5),

$$2u = (1+\vartheta^2\mu^2)^{\frac{1}{2}} + (1+\vartheta\mu^2)^{\frac{1}{2}},$$

$$2\mu v = (\mu^2+\vartheta)^{\frac{1}{2}} + (\mu^2+\vartheta^2)^{\frac{1}{2}}.$$

Hence, eliminating μ, we get

$$\{16u^2v^2 - 8(u^2+v^2)+3\}^2 = 16(u^4+v^4-u^2v^2) - 12(u^2+v^2)+9 \ldots (6),$$

which represents a curve of the eighth order. It may be observed that the locus of the points dividing in a given ratio $\frac{l}{m}$ the points of contact of the parallel tangents, is also of the eighth order. From what we have shown, then, we see that, if l and m are numbers, the arc of such a curve can be expressed as a single elliptic integral of the first kind whose modulus is equal to $\surd(\frac{1}{2})$.

11. I mention here a class of curves from each of which an infinite number of other curves can be generated whose arcs are exactly equal to corresponding arcs of the particular curve considered. Putting $x+iy = u$, $x-iy = v$, where x, y are rectangular coordinates, let us consider the curve whose equation can be expressed in the form

$$\phi(u)\,\phi(v) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

then, if we have

$$u' = \int \{\phi(u)\}^n du,$$

$$v' = \int \{\phi(v)\}^n dv,$$

where u, v are the coordinates of a point lying on the curve (7), it is evident that the point u', v' will describe a curve whose arc is equal to the corresponding arc of (7); for we have

$$du' = \{\phi(u)\}^n du, \quad dv' = \{\phi(v)\}^n dv,$$

and therefore, from (7),

$$du'\,dv' = du\,dv, \quad \text{or} \quad ds'^2 = ds^2.$$

For a considerable number of curves included in (7) the generated curves will be algebraic; as, for instance, the curves whose polar equations are

$$r^m = a^m \cos m\vartheta,$$

being capable of being written in the form

$$(1-u^m)(1-v^m) = 1,$$

the generated curves will be always algebraic, if we suppose n to be a positive integer. For the lemniscate $m = 2$, and, putting

$$u = \frac{2z}{z^2+1}, \quad v = \frac{2iz}{z^2-1}$$

for a point on the curve, we have for the generated curve

$$u' = 2\int \frac{(1-z^2)^{2n+1}dz}{(1+z^2)^{2n+2}}, \quad v' = -2i\int \frac{(1+z^2)^{2n+1}dz}{(1-z^2)^{2n}} \quad \ldots\ldots\ldots(8),$$

which give the coordinates of a point on the curve in terms of a parameter. I shall show now that this curve belongs to Serret's system; for his general curves are given by the equation

$$x+iy = Ce^{i\omega}\int \frac{(z-a)^m(z+a)^n}{(z-a)^{m+1}(z+a)^{n+1}}dz, \quad x-iy = \&c.$$

(see *Liouville*, t. x., p. 267); but, putting $m = n$ in these equations we find $a^2+a^2 = 0$ as the necessary condition that the curve should be algebraic. Hence we see that the curves (8) belong to Serret's system, and correspond to the case of $m = n$.

In the same way we can generate, from the curves whose polar equations are

$$r^3 = a^3\cos 3\vartheta, \quad r^{\frac{1}{2}} = a\cos\left(\frac{3\vartheta}{2}\right),$$

an infinite number of curves whose arcs can be expressed in the same manner, that is, are expressible by elliptic integrals of the first kind with certain numerical moduli. These curves, it may be observed, do not appear to belong to Serret's system. As a further example, we may notice the curves derived from the ellipse of Cassini. The equation of this curve can be written in the form

$$(1-u^2)(1-v^2)-k^4 = 0,$$

where $$u = x+iy, \quad v = x-iy;$$

hence, for a point on the derived curve, we may take

$$k^{2n}u' = \int (1-u^2)^n du, \quad k^{2n}v' = \int (1-v^2)^n dv.$$

Thus, from the known property of the ellipse of Cassini, the arcs of

these curves are expressible as the sum of two elliptic integrals of the first kind with complementary moduli.

12. I now proceed to consider the arc of the general twisted unicursal curve. If for every point of the curve we have

$$x : y : z : 1 = f_1 : f_2 : f_3 : f,$$

where f_1, &c. are binary quantics of the n^{th} degree in a parameter ϑ, and the arcs are rectangular, we obtain

$$ds = \frac{\sqrt{\phi}}{f^2}\, d\vartheta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

in which $$\phi \equiv J_1^2 + J^2 + J_3^2,$$

where J_1, J_2, J_3 are the respective Jacobians of the pairs of quantics f_1, f; f_2, f; f_3, f. Now the curves which I intend to consider more particularly are those of the $(2m)^{\text{th}}$ and $(2m+1)^{\text{th}}$ degrees which intersect the imaginary circle at infinity Ω in $2m$ points. In these cases the expression (9) will admit of considerable simplification; for the roots of ϕ evidently determine the points on the curve at which the tangents intersect Ω, and if $2m$ of these points lie on Ω, ϕ must be divisible by f^2 for $n = 2m$, and by all the squares of the linear factors of f but one for $n = 2m+1$. This may be readily inferred from the fact that the section of the circumscribed developable by a plane has the points where the plane meets the curve for cusps. For the circular curve, then, of the $(2m)^{\text{th}}$ degree, we have

$$ds = \sqrt{\psi}\, \frac{d\vartheta}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where ψ is a binary quantic in ϑ of the degree $4(m-1)$. Now, ψ and f together have $4m-4+2m-3 = 6m-7$ independent absolute invariants, namely, the number of independent invariants less by unity; for the number of the independent invariants of two quantics of the degrees p, q is $$p+q-2.$$

Now, the general unicursal curve of the n^{th} degree has $4n-15$ absolute invariants; for, if the curve be referred to four stationary planes, which are fundamental planes of the curve, each coordinate will be expressible in the form

$$(\vartheta - a)^4 f_{n-4},$$

so that we thus have a biquadratic and four quantics of the degree $n-4$, and these involve $4n-15$ absolute invariants. Also a plane,

namely, in the case we are considering, that at infinity, contains three constants, and $2m-5$ conditions are satisfied by $2m$ points lying on a conic in a given plane, namely Ω. Thus there are

$$8m-15+3-(2m-5) = 6m-7$$

absolute invariants, which, as we have seen, is the same number as that involved in (10). Hence we see that the integral

$$\int \frac{\surd\psi \, d\vartheta}{f},$$

where ψ and f are general binary quantics in ϑ of the degrees $4m-4$ and $2m$, respectively, can be represented by the arc of a twisted curve, with this limitation, however, that all the linear factors of ψ and f must be imaginary. For the curve of the $(2m+1)^{\text{th}}$ degree meeting Ω in $2m$ points, we may put $f = \chi_{2m}\chi_1$, where χ_{2m} gives the $2m$ points on Ω, and χ_1 gives the single real point of the curve at infinity. Hence, since ϕ is divisible by χ_{2m}^2, we have in this case

$$ds = \frac{\surd\psi \, d\vartheta}{\chi_{2m}\chi_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where ψ is of the degree $4m$. Now, this curve, subject to the condition of meeting Ω in $2m$ points, has $6m-3$ absolute invariants; and ψ, χ_{2m}, χ_1, if they were the most general expressions of their kind, would have $6m-2$ absolute invariants. We see thus that ψ, χ_{2m}, χ_1 are connected by a single invariant relation, and, therefore, that the integral (11) cannot be represented by the arc of a curve unless this relation is satisfied.

13. We may now proceed to consider each curve of given degree more particularly. For the circular twisted cubic, we have

$$ds = \frac{\surd\psi \, d\vartheta}{\chi_2\chi_1^2},$$

where ψ is of the fourth degree in ϑ. Now, in order that we may find the relation connecting ψ, χ_2, χ_1 in this case, we refer the cubic to the imaginary tetrahedron formed by an arbitrary point on the curve and the three points at infinity, when we can write

$$x : y : z : u = (t-\alpha)^{-1} : (t-\beta)^{-1} : (t-\gamma)^{-1} : 1.$$

We then easily find

$$x : y : z = (t-\alpha)^{-2} : (t-\beta)^{-2} : (t-\gamma)^{-2} \ldots\ldots\ldots\ldots (12),$$

for the point where a tangent meets the plane u; but, since Ω passes through the points xu, yu, there is

$$\Omega \equiv cz^2+fyz+gzx+hxy.$$

Hence, substituting for x, y, z from (12) in Ω, and dividing by $(t-\alpha)^2(t-\beta)^2$, that is χ_2^2, we get

$$\psi = (t-\gamma)^2\{f(t-\alpha)^2+g(t-\beta)^2+h(t-\gamma)^2\}+c(t-\alpha)^2(t-\beta)^2;$$

from which it follows that $t-\gamma$, that is χ_1, is a factor of the Jacobian of ψ and χ_2. If we suppose then γ infinite, and put $\chi_2 = 1+t^2$, which is evidently permissible, we get

$$ds = \left(t^2+\frac{at^2+bt+c}{1+t^2}\right)\frac{dt}{\sqrt{\psi}},$$

where

$$\psi = t^4+(a+1)t^2+bt+c.$$

The first part of this integral for s, namely,

$$\int \frac{t^2\,dt}{\sqrt{\psi}},$$

evidently depends upon algebraic, circular, and elliptic functions of the three different kinds. The other part of the integral

$$\int \frac{at^2+bt+c}{1+t^2}\,\frac{dt}{\sqrt{\psi}}$$

will be expressed by two elliptic integrals of the third kind with complementary imaginary parameters, which, as has been shown by Legendre in his *Traité des Fonctions Elliptiques*, t. 1, depend upon two similar integrals, one with a circular and the other with a logarithmic parameter.

If in the above we write

$$\psi \equiv \{(t-\alpha)^2+\beta^2\}\{(t-\alpha')^2+\beta'^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

we have $\alpha' = -\alpha$; and then, if the single condition $\alpha = 0$ is satisfied, there is

$$s = \int\left\{t^2+\frac{\beta^2\beta'^2+(\beta^2+\beta'^2-1)\,t^2}{1+t^2}\right\}\frac{dt}{\sqrt{\{(t^2+\beta^2)(t^2+\beta'^2)\}}} \ldots (14),$$

in which, putting $t = \beta\tan\phi$, we get the arc, minus an algebraic part, equal to

$$lE(\phi)+m\Pi(n,\phi).$$

If, in (13), $\beta' = \beta$, and then we put

$$t = \sqrt{(\alpha^2+\beta^2)}\tan\tfrac{1}{2}\phi,$$

we see that the arc in this case also involves a single elliptic integral of the third kind.

14. Proceeding now to the case of the circular twisted quartic, we have

$$s = \int \frac{\sqrt{\psi}\,d\vartheta}{f} = \int \frac{\psi}{f}\,\frac{d\vartheta}{\sqrt{\psi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

where ψ and f are general binary quantics with imaginary roots. If, then, $\vartheta^2+p\vartheta+q$, $\vartheta^2+p'\vartheta+q'$ are the real factors of f, we may write

$$s = \int \left\{ \frac{a\vartheta^2+b\vartheta+c}{\vartheta^2+p\vartheta+q} + \frac{a'\vartheta^2+b'\vartheta+c'}{\vartheta^2+p'\vartheta+q'} \right\} \frac{d\vartheta}{\sqrt{\psi}}.$$

Hence from the result of Legendre, referred to above, we see that in the expression for the arc there are involved four elliptic integrals of the third kind with different parameters, two of which are circular and two logarithmic. I shall now transform this integral (15) into another in which the consideration of particular cases becomes easier. Let P, Q be the two real quadratic factors of ψ; then, if J is the Jacobian of P, Q, it is evident that a, b, h, l, m can be determined so that

$$aP^2+bQ^2+2hPQ+J(lP+mQ)$$

should represent any given binary quartic; and this is the form which I shall assume for f. We have then

$$s = \int \frac{PQ}{aP^2+bQ^2+2hPQ+J(lP+mQ)}\,\frac{d\vartheta}{\sqrt{PQ}} \quad \ldots\ldots\ldots(16);$$

hence, putting $P = Qz$, and observing that, by a known theorem, we may assume

$$J^2 = -(P-\lambda_1 Q)(P-\lambda_2 Q),$$

in which λ_1, λ_2 are real quantities, we obtain

$$s = \int \frac{z}{\{az^2+2hz+b+(lz+m)\,\Delta\}}\,\frac{dz}{\Delta\sqrt{z}},$$

where

$$\Delta^2 = (\lambda_1-z)(z-\lambda_2).$$

In the general case, then, there is

$$s = \int \frac{(az^2+2hz+b)}{\{(az^2+2hz+b)^2-(lz+m)^2\,\Delta^2\}}\,\frac{z\,dz}{\Delta\sqrt{z}} - \int \frac{(lz+m)\sqrt{z}\,dz}{\{(az^2+2hz+b)^2-(lz+m)^2\,\Delta^2\}} \quad \ldots\ldots\ldots(17),$$

the latter integral evidently depending on circular and logarithmic functions. Putting then

$$\phi(z) = \sqrt{\{z\,(\lambda_1-z)(z-\lambda_2)\}},$$

we may express $\dfrac{z\,(az^2+2hz+b)}{(az^2+2hz+b)^2-(lz+m)^2\,\Delta^2}$

in the form $\dfrac{m_1\sqrt{\phi(\alpha)}}{z-\alpha}+\dfrac{m_2\sqrt{\phi(\beta)}}{z-\beta}+\dfrac{m_3\sqrt{\phi(\gamma)}}{z-\gamma}+\dfrac{m_4\sqrt{\phi(\delta)}}{z-\delta}$,

where α, β; γ, δ; m_1, m_2; m_3, m_4 are supposed to be conjugate imaginaries. Thus the first part of the expression (17) for the arc becomes

$$\tfrac{1}{2}(m_1+m_2)\int\left\{\frac{\sqrt{\phi(\alpha)}}{z-\alpha}+\frac{\sqrt{\phi(\beta)}}{z-\beta}\right\}\frac{dz}{\sqrt{\phi(z)}}$$

$$+\tfrac{1}{2}(m_1-m_2)\int\left\{\frac{\sqrt{\phi(\alpha)}}{z-\alpha}-\frac{\sqrt{\phi(\beta)}}{z-\beta}\right\}\frac{dz}{\sqrt{\phi(z)}}$$

$$+\tfrac{1}{2}(m_3+m_4)\int\left\{\frac{\sqrt{\phi(\gamma)}}{z-\gamma}+\frac{\sqrt{\phi(\delta)}}{z-\delta}\right\}\frac{dz}{\sqrt{\phi(z)}}$$

$$+\tfrac{1}{2}(m_3-m_4)\int\left\{\frac{\sqrt{\phi(\gamma)}}{z-\gamma}-\frac{\sqrt{\phi(\delta)}}{z-\delta}\right\}\frac{dz}{\sqrt{\phi(z)}} \quad \ldots\ldots\ldots(18).$$

Now, by the theory of the elliptic integral of the third kind [that is, by means of the result

$$\Pi(\alpha, u) = \Pi(u, \alpha)+u\mathrm{Z}(\alpha)-\alpha\mathrm{Z}(u),$$

and by means of the addition formula for these integrals], each of the four integrals in (18) can be expressed as a real Π function. We may put, in fact,

$$\int\left\{\frac{\sqrt{\phi(\alpha)}}{z-\alpha}+\frac{\sqrt{\phi(\beta)}}{z-\beta}\right\}\frac{dz}{\sqrt{\phi(z)}} = \int\frac{\sqrt{\phi(p)}}{z-p}\,\frac{dz}{\sqrt{\phi(z)}}+\lambda\vartheta+\mu\int\frac{dz}{\sqrt{\phi(z)}},$$

$$\int \left\{ \frac{\sqrt{\phi(\alpha)}}{z-\alpha} - \frac{\sqrt{\phi(\beta)}}{z-\beta} \right\} \frac{dz}{\sqrt{\phi(z)}}$$

$$= i \int \frac{\sqrt{-\phi(q)}}{z-q} \frac{dz}{\sqrt{\phi(z)}} + i\lambda'\phi + i\mu' \int \frac{dz}{\sqrt{\phi(z)}} \ldots (19),$$

where ϑ is a logarithm and ϕ a circular function, and $i^2 = -1$.

15. We can now easily find the conditions that the arc should involve a lesser number of different integrals of the third kind. Since

$$\int \left\{ \frac{m\sqrt{\phi(p)}}{z-p} + \frac{n\sqrt{\phi(p')}}{z-p'} \right\} \frac{dz}{\sqrt{\phi(z)}},$$

where m, n are numbers, and p, p' are real quantities, depends upon a single integral, we see that the arc can be made to involve but three distinct integrals, if one of the quantities

$$\frac{m_1 \pm m_2}{m_3 \pm m_4}$$

is commensurable, and that it will involve two integrals only if both these quantities are commensurable. A simple example is obtained by taking

$$m_1 \pm m_2 = 0, \quad \text{or} \quad m_3 \pm m_4 = 0,$$

in which case, it may be easily shown, the relation between f and ψ is

$$f'(\alpha)\sqrt{\psi(\beta)} - f'(\beta)\sqrt{\psi(\alpha)} = 0,$$

where α, β are two conjugate imaginary roots of f.

16. There is another case in which the number of the integrals is reduced, and which does not come under the preceding. Suppose

$$\phi(p) = 0,$$

or, which comes to the same thing, putting

$$z = \lambda_1 \cos^2\phi + \lambda_2 \sin^2\phi = \lambda_1 \Delta^2 amu, \text{ say},$$

and $$\alpha = \lambda_1 \operatorname{cn}^2 a_1 + \lambda_2 \operatorname{sn}^2 a_1, \quad \beta = \lambda_1 \operatorname{cn}^2 a_2 + \lambda_2 \operatorname{sn}^2 a_2,$$

if we have $$a_1 + a_2 = K,$$

then, since $\Pi(K-\alpha, u)$ can be expressed in terms of $\Pi(\alpha, u)$, we see that only three distinct integrals are involved. In this case, a real quadratic factor of f is harmonically connected with the Jacobian of

the two real quadratic factors of ψ, that is, in (16), $lP+mQ$ is a factor of aP^2+bQ^2+2hPQ. It is easy to see then, at once, that the expression for s (17) only involves three different integrals.

17. Again, if f have two factors of the form $\lambda P+\mu Q$, l, m, both vanish in (16), and we get

$$s=\int \frac{z}{az^2+2hz+b}\ \frac{dz}{\sqrt{\{z(\lambda_1-z)(z-\lambda_2)\}}} \quad \ldots\ldots\ldots\ldots(20).$$

Hence, in the latter case, the arc is expressed by two elliptic integrals of the third kind. Furthermore, if f is of the form $\psi+\lambda H$, where H is the Hessian of ψ, we can show that the arc depends upon a single elliptic integral of the third kind. We have

$$s=\int \frac{\psi}{\psi+\lambda H}\ \frac{d\vartheta}{\sqrt{\psi}}=\int \frac{1}{1+\lambda z}\ \frac{d\vartheta}{\sqrt{\psi}},$$

putting $H=\psi z$, according to Weierstrass's transformation; but

$$\psi^2 dz=\psi dH-H d\psi=J d\vartheta,$$

where J is the Jacobian of ψ and H. Now, by the theory of the binary quartic, J^2 is proportional to

$$-(4H^3-SH\psi^2+T\psi^3)=-\psi^3(4z^3-Sz+T),$$

where S and T are the invariants of ψ. Hence we have

$$\frac{d\vartheta}{\sqrt{\psi}}=\frac{dz}{\sqrt{-(4z^3-Sz+T)}},$$

and

$$s=\int \frac{1}{1+\lambda z}\ \frac{dz}{\sqrt{-(4z^3-Sz+T)}} \quad \ldots\ldots\ldots\ldots(21),$$

which is of the form mentioned above. I have already considered the rectification of the curve in this case (see *Proceedings of the London Mathematical Society*, Vol. XVII., p. 39), and have shown that the arc is exactly equal to the arc of a sphero-conic. This sphero-conic is, however, not necessarily real, in which case the integral has a logarithmic parameter, so that the arc of the curve we are considering represents the elliptic integral of the third kind in the most general manner, which the arc of the sphero-conic does for integrals with circular parameters only. The particular case noticed *loc. cit.*, in which the arc of the curve is a pure elliptic integral of the first kind, evidently arises from making f and ψ coincide. It is easy to see, then, that Ω touches the section by the plane at infinity of the circumscribed developable at its four cusps, namely, the four points where

the plane at infinity meets the curve, or, which is the same thing, the osculating planes to the curve at the four points on Ω touch Ω.

18. We may notice, also, two simple cases in which the expression (17) is made to depend upon three integrals. Suppose f and ψ are such that $f \equiv \psi + R^2$, identically, where R is a quadratic; then, by a particular case of Abel's theorem, there is $\alpha_1 \pm \alpha_2 + \alpha_3 \pm \alpha_4 = \omega$, where α_1, &c. are the arguments of the parameters, from which the result stated follows at once. In fact, there is then either $p = p'$ or $q = q'$, where p, q refer to α, β, and p', q' to γ, δ, in (19). Again, if P, Q are the real quadratic factors of ψ, and f is of the form $Px^2 + Qy^2$, where x, y are linear expressions in the variable, then, by another particular case of Abel's theorem (see Legendre's *Traité des Fonctions Elliptiques*, t. III., p. 192), there is $\alpha_1 \pm \alpha_2 + \alpha_3 \pm \alpha_4 = \frac{1}{2}\omega$, from which the dependence on three integrals becomes manifest. These results would also follow at once from the respective identities,

$$\int \frac{J(R, \psi)\, d\vartheta}{(\psi + R^2)\sqrt{\psi}} = \tan^{-1}\left(\frac{\sqrt{\psi}}{R}\right),$$

$$\int \frac{\{PQ + xyJ(P, Q)\}}{Px^2 + Qy^2} \frac{d\vartheta}{\sqrt{PQ}} = \tan^{-1}\left\{\frac{x}{y}\sqrt{\left(\frac{P}{Q}\right)}\right\} \ldots\ldots(22);$$

for by means of them we should evidently obtain a linear relation connecting the four integrals involved in each case.

19. Let us now consider the case in which Ω is touched by the osculating planes at two of the points where it meets the curve. It is evident then that f and ψ have a common quadratic factor P, say. Hence, if $f = PR$, we have

$$s = \int \frac{Q}{R} \frac{d\vartheta}{\sqrt{PQ}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23),$$

which is expressible by means of two elliptic integrals of the third kind, one of which has a circular and the other a logarithmic parameter. Furthermore, if we suppose R to be of the form $lP + mQ$, and put $P = Qz$, we get

$$s = \int \frac{1}{lz + m} \frac{dz}{\sqrt{\{z(\lambda_1 - z)(z - \lambda_2)\}}} \quad \ldots\ldots\ldots\ldots(24),$$

which is an elliptic integral of the third kind with a parameter either circular or logarithmic. The arc, then, of this curve also gives a very general representation of such integrals. It may be observed that if,

in this case, we take $m = 0$, or suppose $f = H$ in (21), we get two curves whose arcs are exactly equal to those of ellipses.

20. The integral $$\int \frac{\sqrt{PQ}}{f}\, d\vartheta$$

would admit of considerable simplification in the case also in which f contained the Jacobian J of P, Q as a factor; but this can never obtain for the arcs of the curves we are considering, for, since P, Q have imaginary roots, their Jacobian J has real roots, and therefore can never be a factor of f, all of whose roots are imaginary.

21. For the curve which has double contact with Ω, we put f equal to the square of a quadratic R^2, say. Now, it is evident that any quadratic R can be expressed in the form

$$lP+mQ+nJ;$$

hence we have $$s = \int \frac{PQ}{(lP+mQ+nJ)^2}\, \frac{d\vartheta}{\sqrt{PQ}}$$

$$= \int \frac{z}{(lz+m+n\Delta)^2}\, \frac{dz}{\Delta\sqrt{z}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25),$$

where $$P = Qz \text{ and } \Delta^2 = (\lambda_1 - z)(z - \lambda_2).$$

In this case, then, the arc involves an algebraic part and an elliptic integral of the second kind, besides two elliptic integrals of the third kind, one with a circular and the other with a logarithmic parameter. If, however, the single condition $n = 0$ is satisfied, there is but one integral of the third kind, whose parameter may be either circular or logarithmic.

22. We may now consider another particular form of the curve, namely, such that the section of the circumscribed developable by the plane at infinity has double contact with Ω at points other than those on the curve. In this case ψ is evidently a perfect square, and there is

$$s = \int \frac{R\, d\vartheta}{PQ},$$

where P, Q are the two real factors of f; hence, if we write R in the form

$$lP+mQ+nJ,$$

we find $$s = l\int \frac{d\vartheta}{Q} + m\int \frac{d\vartheta}{P} + n \log\left(\frac{P}{Q}\right) \quad \ldots\ldots\ldots\ldots\ldots (26),$$

that is, the arc is expressible by two circular arcs and a logarithmic function. If we put $n = 0$, and suppose $l : m$ commensurable, the arc of the curve, we see, will represent a circular function.

23. Passing by the case of the quintic, which does not seem to be of much interest, we may proceed at once to consider the circular sextic. For this curve, we have

$$s = \int \frac{\psi}{f} \frac{d\vartheta}{\sqrt{\psi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

where ψ and f are general expressions in ϑ of the eighth and sixth degrees, respectively. Let us suppose, now, that the osculating planes of the curve at the six points on Ω are touched by Ω, then ψ will evidently contain f as a factor. Hence, if $\psi_1, \psi_2, \psi_3, \psi_4$ are the four real quadratic factors of ψ, we may take

$$f = \psi_1 \psi_2 \psi_3,$$

and we have

$$s = \int \frac{\psi_4 \, d\vartheta}{\sqrt{(\psi_1 \psi_2 \psi_3 \psi_4)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

which evidently gives a very general representation of the first kind of the second class of hyper-elliptic integrals. Further, if we suppose $\psi_1, \psi_2, \psi_3, \psi_4$ to belong to a system in involution, these quadratics can be transformed linearly so as to involve the square of ϑ only, that is,

we may take $\psi_1 = \vartheta^2 + a^2$, &c.;

hence, putting $\vartheta^2 = z, \quad a^2 = a$, &c.,

we have

$$s = \int \frac{(z-d)\, dz}{\sqrt{\{z(z+a)(z+b)(z+c)(z+d)\}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

which gives a very general representation of the first kind of the first class of hyper-elliptic integrals. Hence, by the theory of Abelian integrals, if we take three variable points A, B, C on the latter curve, the condition

$$\text{arc } OA \pm \text{arc } OB \pm \text{arc } OC = \text{a constant},$$

where O is a fixed point, is consistent with two algebraic relations. Also, for the former curve, if we take a fourth point, a similar relation connecting the arcs will be consistent with three algebraic relations.

24. It may be observed that, if we have $ab = cd$ in (29), and if we

put, then, $$z+\left(\frac{ab}{z}\right)=2y,$$

we find that the arc consists of the sum of two elliptic integrals of the first kind with different moduli. We may notice, also, a general case in which the arc of the sextic depends upon elliptic integrals. Suppose the section of the circumscribed developable by the plane at infinity to have double contact with Ω, then ψ assumes the form $\phi^2\chi$, where ϕ is a quadratic, and χ a quartic; we have then

$$s=\int\frac{\phi\chi}{f}\frac{d\vartheta}{\sqrt{\chi}} \quad \text{.....(30),}$$

which evidently depends, in general, on six elliptic integrals of the third kind, with different parameters and the same modulus. By considering special cases, several of these different parameters may coincide until the arc is expressed by a single elliptic integral of the third kind.

25. From the fact that the arcs of curves are in general expressed by Abelian integrals, we should be led, from the properties of those functions, to expect certain geometrical theorems depending on the comparison of different arcs. That is, in the case of elliptic and hyper-elliptic integrals of the first kind, we should expect simple geometrical relations connecting the extremities of arcs whose algebraic sums were given; and, in the case of the integrals of the second and third kind, we should expect the sums of arcs connected with algebraic or circular quantities, respectively. The only important result of this nature seems to be the theorem which is called Dr. Graves's, and its extension to the lines of curvature of a quadric. This theorem was originally obtained by Dr. Graves for the case of the sphero-conic (see his "Translation of Chasles' two Memoirs on Cones of the Second Degree and Spherical Conics," Dublin, 1841, p. 77, where the result is given as the reciprocal of another theorem, namely, if two spherical conics have the same cyclic arcs, any arc touching the inner curve will cut off from the outer one a segment of constant area). An extension of Dr. Graves's theorem was discovered by MacCullagh, and both results were shortly afterwards given independently by Chasles (*Comptes Rendus*, t. XVII., 1843, pp. 838—844), who added some properties also connecting circles with elliptic and hyperbolic arcs whose difference is rectifiable. Several properties of equal arcs of the lemniscate of Bernoulli were thence derived by Chasles (*Comptes Rendus*, t. XXV., 1845, pp. 190—201).

26. It has been remarked that the theorems alluded to above were not anticipated by those who had applied the integral calculus to the rectification of conic sections; but this, indeed, could hardly have been expected to have been the case, as the results flow from a simple geometrical property of the conic sections. In fact, the geometrical method would, in general, infer theorems from some particular property, and then consider whether there was any known curve that possessed this property. The general theorem, of which Dr. Graves's result is the particular application to confocal conics, was originally given by Leibnitz in a letter dated 3rd Jan., 1704, addressed to John Bernoulli, as has been pointed out by Mr. Samuel Roberts (see *Proceedings of the London Mathematical Society*, Vol. XII., p. 121).

27. I purpose now to give a few simple properties of arcs of certain curves included in the form

$$r^m = a^m \cos m\vartheta,$$

where r, ϑ are polar coordinates. Considering the cubic corresponding to the case $m = -3$, we have the arc proportional to

$$\int \frac{z\,dz}{\sqrt{(z^3-1)}},$$

where $$r^2 = a^2 z.$$

Now, eliminating ϑ between the equation of the curve and the line

$$r\cos(\vartheta-\omega)-p=0,$$

we get $$(4p^3-a^3\cos 3\omega-3pr^2)^2-\sin^2 3\omega\,(r^6-a^6)=0 \quad \ldots\ldots\ldots(31),$$

which may be written

$$(\alpha+\beta z)^2-(z^3-1)\equiv\phi(z), \text{ say } =0 \quad \ldots\ldots\ldots\ldots\ldots(32).$$

But, differentiating this relation, we have

$$\phi'(z)\,dz+2(\alpha+\beta z)(d\alpha+z\,d\beta)=0,$$

or, from (32), for a root of ϕ,

$$\frac{dz}{\sqrt{(z^3-1)}}+\frac{2(d\alpha+z\,d\beta)}{\phi'(z)}=0,$$

whence we get $$\Sigma\frac{dz}{\sqrt{(z^3-1)}}=0, \text{ identically,}$$

and $$\Sigma\frac{z\,dz}{\sqrt{(z^3-1)}}=0,$$

if

$$d\beta = 0,$$

that is, if β is constant. Hence, from (31), we see that tangents to the tricuspidal hypocycloid

$$p = b \sin 3\omega$$

meet the curve in three points, the algebraic sum of whose distances, measured along the curve from fixed points, is constant. We say fixed points, and not a fixed point, because collinear points are situated on different branches of the curve.

In the same way, if we seek the intersection of a circle with the curve, we get an equation of the form

$$z^3 - 1 - (\alpha + \beta z + \gamma z^2 + \delta z^3)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

whence, by Abel's theorem, we get for the six roots

$$\Sigma \frac{dz}{\sqrt{(z^3-1)}} = 0, \quad \Sigma \frac{z\,dz}{\sqrt{(z^3-1)}} = 0,$$

from which we infer that, if a circle meet the curve in six real points, the algebraic sum of the arcs, measured from these six points to fixed points on the curve, is constant. The same, it may be observed, will also be true for a conic having the origin for centre, or, which is the same thing, if the tangents at the six points are touched by a circle.

28. Let us consider now the inverse of the curve mentioned in the preceding article, that is, the curve whose polar equation is

$$r^3 = a^3 \cos 3\vartheta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

In this case the arc is proportional to

$$\int \frac{dz}{\sqrt{\{z(z^3-1)\}}}.$$

But, putting z^{-1} for z in this expression, we get the integral which has been called the argument of a point on the inverse cubic curve. Hence, by inverting the known properties of points on a cubic curve, we can arrive at certain results involving arcs of the curve (34). Thus, if a circle be described through the origin to meet the curve in three points, the sum of the arcual distances of these points from the origin is equal to the perimeter of one of the ovals.

Again, we find that two real circles can be described through the origin and a point on the curve to touch the curve elsewhere, and that the arc joining the points of contact is equal to half the perimeter of an oval. Also we may deduce that, if a circle intercept three

arcs on the curve, three properly selected points of bisection of these arcs will lie on a circle passing through the origin.

29. I may notice here a certain locus connected with the lemniscate, namely, that of the centre of gravity of an arc of given length. Let the curve be written

$$r^2 = a^2 \cos 2\vartheta,$$

then

$$ds = \frac{a\,d\vartheta}{\sqrt{\cos 2\vartheta}},$$

and, if x, y are the coordinates of the centre of gravity of an arc of length l, we find

$$lx = a^2(\sin\vartheta_1 - \sin\vartheta_2), \quad ly = a^2(\cos\vartheta_2 - \cos\vartheta_1),$$

whence $l^2(x^2+y^2) = 4a^4 \sin^2\tfrac{1}{2}(\vartheta_1-\vartheta_2)$, $\tan\tfrac{1}{2}(\vartheta_1+\vartheta_2) = y/x$ ...(35).

Now, since the arc is of given length, there is a relation of the form

$$m\cos\vartheta_1\cos\vartheta_2 + n\sin\vartheta_1\sin\vartheta_2 = 1,$$

where

$$m^2+n^2 = 2.$$

We thus get, for the equation of the locus, from (35),

$$l^2(m+n)(x^2+y^2)^2 - 4a^4\{(m-1)x^2+(n-1)y^2\} = 0 \ldots\ldots(36),$$

which represents the inverse of a concentric ellipse.

30. I shall now consider some more general relations connecting the arcs of curves; and, for this purpose, we must have a general method of comparing transcendents. I commence, therefore, by demonstrating a general theorem concerning the points of intersection of a conic with a given curve.

Let

$$U \equiv f(x, y) = 0$$

be the equation of a curve of the n^{th} degree in Cartesian coordinates, and let us take

$$x : y : 1 = f_1 : f_2 : f \ldots\ldots\ldots\ldots\ldots\ldots(37)$$

for a point on a conic, where f_1, f_2, f are quadratic expressions in a parameter ϑ. Substituting, then, the coordinates of a point on the conic in $U = 0$, we get, to determine the $2n$ points of intersection,

$$f^n U \equiv \phi(\vartheta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(38),$$

where $\phi(\vartheta)$ is a rational integral expression in ϑ of the $(2n)^{\text{th}}$ degree.

Considering now the roots of

$$\phi(\vartheta) = 0$$

as functions of the constants which occur in the equation of the conic, that is, in f_1, f_2, f, we get, by differentiation,

$$\phi'(\vartheta)\, d\vartheta + f^n \delta U = 0,$$

or
$$\phi'(\vartheta)\, d\vartheta + f^n \left\{ \frac{dU}{dx}\, \delta\left(\frac{f_1}{f}\right) + \frac{dU}{dy}\, \delta\left(\frac{f_2}{f}\right) \right\} = 0 \quad \ldots\ldots\ldots\ldots (39),$$

where δ represents differentiation with regard to the constants just referred to. But, calling u the argument of a point on U, we have

$$\frac{dx}{\frac{dU}{dy}} = -\frac{dy}{\frac{dU}{dx}} = du.$$

Hence, from (39), we may write

$$\phi'(\vartheta)\, du + f^n \left\{ \left(\frac{dx}{d\vartheta}\right) \delta\left(\frac{f_2}{f}\right) - \left(\frac{dy}{d\vartheta}\right) \delta\left(\frac{f_1}{f}\right) \right\} = 0 \quad \ldots\ldots\ldots (40).$$

Now $\dfrac{dx}{d\vartheta} = \dfrac{d}{d\vartheta}\left(\dfrac{f_1}{f}\right) + \dfrac{\delta}{d\vartheta}\left(\dfrac{f_1}{f}\right)$, $\dfrac{dy}{d\vartheta} = \dfrac{d}{d\vartheta}\left(\dfrac{f_2}{f}\right) + \dfrac{\delta}{d\vartheta}\left(\dfrac{f_2}{f}\right)$;

and
$$\frac{d}{d\vartheta}\left(\frac{f_1}{f}\right) \delta\left(\frac{f_2}{f}\right) - \frac{d}{d\vartheta}\left(\frac{f_2}{f}\right) \delta\left(\frac{f_1}{f}\right)$$

$$= \frac{J_2}{f^4}(f\delta f_2 - f_2 \delta f) + \frac{J_1}{f^4}(f\delta f_1 - f_1 \delta f)$$

$$= \frac{1}{f^3}(J_1 \delta f_1 + J_2 \delta f_2) - \frac{\delta f}{f^4}(J_1 f_1 + J_2 f_2) \quad \ldots\ldots\ldots\ldots (41),$$

where J_1, J_2, J_3 are the Jacobians of f_2, f; f, f_1; f_1, f_2, respectively.

But we have
$$J_1 f_1 + J_2 f_2 + J_3 f_3 = 0,$$

identically; hence (41) becomes

$$\frac{1}{f^3}(J_1 \delta f_1 + J_2 \delta f_2 + J_3 \delta f).$$

Thus, finally, we have, from (40),

$$du + \frac{f^{n-3} G}{\phi'(\vartheta)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42),$$

where
$$G \equiv J_1 \delta f_1 + J_2 \delta f_2 + J_3 \delta f_3,$$

or
$$\begin{vmatrix} \delta f_1, & \delta f_2, & \delta f \\ \frac{df_1}{d\lambda}, & \frac{df_2}{d\lambda}, & \frac{df}{d\lambda} \\ \frac{df_1}{d\mu}, & \frac{df_2}{d\mu}, & \frac{df}{d\mu} \end{vmatrix},$$

if f_1, f_2, f are written as homogeneous functions of λ, μ. I now mention the geometrical meaning of $G = 0$. Suppose the conic to vary subject to four conditions, then it will evidently, in general, touch an envelope in four points. These points are found from $\delta V = 0$, where V is the equation of the conic; hence, from (37), we easily find

$$\frac{d}{d\vartheta}\left(\frac{f_1}{f}\right)\delta\left(\frac{f_2}{f}\right) - \frac{d}{d\vartheta}\left(\frac{f_2}{f}\right)\delta\left(\frac{f_1}{f}\right) = 0.$$

We thus see that $G = 0$ gives the four points where the conic is intersected by a consecutive.

31. I shall now notice the principal applications of this result. Let U be a conic, then we have

$$du + \frac{G}{f\phi'(\vartheta)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43).$$

Now, if the variable conic have double contact with a fixed conic S, we may put

$$G = G_1 f \surd S,$$

where G_1 is a quadratic factor of G; for if G_2 be the other factor of G corresponding to the points of contact with S, we have

$$G_2 = fP,$$

where P is the chord of contact, and $P^2 = S$ is the equation of the variable conic. We obtain thus

$$\frac{du}{\surd S} + \frac{G_1}{\phi(\vartheta)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44),$$

whence, for the four points of intersection, we get

$$\Sigma_4^1 \frac{du}{\surd S} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45).$$

Again, suppose the variable conic to touch a quartic curve W in four points, then, in the same way, we may put

$$G = f^2 \surd W,$$

whence we get $$\frac{du}{\sqrt{W}} + \frac{f}{\phi'(\vartheta)} = 0;$$

we have then $$\Sigma \frac{du}{\sqrt{W}} = \Sigma \frac{x\,du}{\sqrt{W}} = \Sigma \frac{y\,du}{\sqrt{W}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (46).$$

Applying the result (42) to the case of the cubic, we find that, if a conic having double contact with a fixed conic S meet the curve, then, for the six points of intersection, we have

$$\Sigma \frac{L\,du}{\sqrt{S}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47),$$

where L is an arbitrary line, besides the known relation

$$\Sigma\, du = 0.$$

If the conic have quartic contact with a quartic curve W, we find

$$\Sigma \frac{V\,du}{\sqrt{W}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48),$$

where V is an arbitrary conic.

For the curves of the fourth order, we find that, if a conic having double contact with a given conic S meet the curve, then

$$\Sigma \frac{V\,du}{\sqrt{S}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49),$$

where V is an arbitrary conic. Also, if the variable conic have quartic contact with a quartic W, then

$$\Sigma \frac{V'\,du}{\sqrt{W}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

where V' is an arbitrary cubic.

32. There is a modification of the preceding results if the variable conic pass through two fixed points on the curve. In the case of the cubic, we have

$$du + \frac{G}{\phi'(\vartheta)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51),$$

where G gives the two points of contact with the envelope, and $\phi(\vartheta)$ involves the four variable points of intersection. Applying this result to the circular cubic and circle, we find that du is proportional to

$$\frac{AP.AQ}{AB.AC.AD} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52),$$

where A, B, C, D are the four points of intersection, and P, Q the two points of contact of the circle with its envelope. The expression (52), it may be remarked, can be replaced by $\Delta_1 p_1$, where Δ_1 is the area of the triangle BCD, and p_1 is the perpendicular from A on the chord of contact. By inversion, then, we find that the preceding result holds for the intersection of a circle with a bicircular quartic. Hence, if a circle have double contact with a bicircular quartic W, we have, for the four points of intersection with a circular cubic or bicircular quartic,

$$\Sigma \frac{L\,du}{\sqrt{W}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

where L is an arbitrary line.

33. I now proceed to deduce some theorems by means of the foregoing results. I first consider particularly the bicircular quartic with a centre. In this case there are two focal conics, which may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad \frac{x^2}{a^2-k^2} + \frac{y^2}{b^2-k^2} - 1 = 0,$$

the corresponding Jacobian circles being

$$x^2+y^2 = \pm k^2,$$

respectively. I may remark here then that, if $a^2+b^2>k^2$, the arc of the curve is equal to the sum or difference of an arc of each of the mutually reciprocal sphero-conics

$$\frac{b^2x^2}{a^2-k^2} + \frac{a^2y^2}{b^2-k^2} - z^2 = 0,$$

$$(a^2-k^2)\frac{x^2}{b^2} + (b^2-k^2)\frac{y^2}{a^2} - z^2 = 0,$$

where $$x^2+y^2+z^2 = a^2+b^2-k^2.$$

These curves are the sections of the sphere by the cylinders standing on the focal conics. This suggests the existence of some theorem connecting the arcs of the curve with those of a circle; and, in fact, I proceed to prove that, supposing the curve to be written in the form

$$SS' = d^4,$$

where S, S' are two equal circles, if a variable circle Σ meet the curve and the two circles S, S', so that the sum of the arcs intercepted on the curve, minus the sum of the arcs intercepted on S, S', is constant, then Σ has double contact with a concentric bicircular quartic. I

demonstrate this result in the converse form, which evidently comes to the same thing. Putting

$$S \equiv (u-c)(v-c)-r^2,$$

$$S' \equiv (u+c)(v+c)-r^2,$$

where $x+iy=u, \quad x-iy=v,$

if $$U \equiv SS'-d^4,$$

we have
$$\begin{aligned}\frac{dU}{du}\frac{dU}{dv} &= S^2(S'+r^2)+S'^2(S+r^2)+2SS'(uv-c^2)\\ &= 4d^4uv+4r^2c^2(u+v)^2\\ &= 4\{d^4(x^2+y^2)+4r^2c^2x^2\}\\ &= 16\{a^2(a^2-k^2)x^2+b^2(b^2-k^2)y^2\};\end{aligned}$$

for $$d^4=4b^2(b^2-k^2), \quad r^2=a^2+b^2-k^2, \quad c^2=a^2-b^2.$$

Now, if du is the argument of a point on the curve

$$\begin{aligned}ds &= \sqrt{\left\{\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2\right\}}\,du\\ &= \sqrt{\{a^2(a^2-k^2)x^2+b^2(b^2-k^2)y^2\}}\,du = P\,du \text{ say...(54).}\end{aligned}$$

Hence, if a circle have double contact with the bicircular quartic

$$a^2(a^2-k^2)x^2+b^2(b^2-k^2)y^2+\lambda U \text{ .....(55),}$$

we have, from (51), $$\frac{du}{P} \propto \frac{f}{\phi'(\vartheta)};$$

therefore, from (54),

$$ds = P\,du \propto f\{d^4uv+r^2c^2(u+v)^2\}/\phi'(\vartheta).$$

Now, for any point on a circle, we may take

$$u : v : 1 = \alpha+\rho t : \beta+\rho t^{-1} : 1 \text{ .....(56).}$$

Hence, for the four points of intersection with the curve, we get

$$\begin{aligned}\Sigma ds &\propto \Sigma \frac{1}{t\phi'(t)}\{d^4t(\alpha+\rho t)(\beta t+\rho)+r^2c^2(\alpha t+\beta t+\rho+\rho t^2)^2\}\\ &= r^2c^2\rho^2\left(1-\frac{1}{t_1t_2t_3t_4}\right) \text{.....(57).}\end{aligned}$$

But, substituting the values (56) of u, v in the equation of the curve, we get $$(\beta^2-c^2)t^4+\ldots+\alpha^2-c^2=0,$$

from which we see that

$$t_1 t_2 t_3 t_4 = \frac{a^2 - c^2}{\beta^2 - c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58).$$

We see thus, from (57) and (58), that the expression for Σds is independent of d^4, and therefore retains the same value when d vanishes, in which case the curve breaks up into the two circles S, S'. Hence the theorem which I have stated above follows at once.

34. If the circle passing through two pairs of inverse points on the curve has double contact with the bicircular quartic

$$(x^2+y^2+k^2)^2 - 4(a^2x^2+b^2y^2) + \frac{4}{\lambda^2}\left\{a^2(a^2-k^2)x^2 + b^2(b^2-k^2)y^2\right\} = 0,$$

it can be easily shown that the generating circles of the curve at the points intersect on a confocal bicircular quartic; that is, in the case which we are considering, the generating circles at the extremities of arcs, connected in the manner mentioned above, intersect on a confocal bicircular quartic.

As a particular case of this result, let $r = 0$, then the curve is an ellipse of Cassini, and we have

$$\Sigma ds = 0.$$

Hence we see that, if generating circles of the ovals of Cassini intercept arcs whose sum or difference is given, then their points of intersection will lie on a confocal bicircular quartic.

Again, suppose the envelope (55) to coincide with the curve, that is, let $\lambda = 0$, and we see that the generating circles of the curve intercept constant arcs on the circles S, S'. In fact, we can easily verify that the circle

$$x^2+y^2-2ax\cos\phi - 2by\sin\phi + k^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (59)$$

cuts off constant arcs from the circles

$$y^2 + (x \pm c)^2 - (a^2+b^2-k^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (60);$$

for the chords of intersection of (59) with (60) touch the circles

$$x^2+y^2 \pm 2cx - b^2 = 0.$$

The theorem given above connects arcs of a certain quartic with those of circles, and is a geometrical consequence of the fact that the arc of the curve depends upon elliptic integrals of the third kind with circular parameters. But, in the case of the integrals having logarithmic parameters, the arcs of the curve cannot be connected

with arcs of circles, from which we may infer that the arcs of the general bicircular quartic cannot be connected with circular arcs; for the expression found at p. 100 involves imaginary parameters, and therefore, by transformation, is expressed by integrals with circular and logarithmic parameters. We may, however, notice the following result for the general curve. If two generating circles of the same system intersect on a confocal bicircular quartic, the sum or difference of the arcs intercepted by the points of contact is rectifiable by logarithmic and circular functions.

35. We may also arrive at some results concerning arcs intercepted on the curve by conics which satisfy certain conditions. Let U be a general bicircular quartic, then we can easily show that

$$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2,$$

where x, y are rectangular coordinates, can, by combination with $U=0$, be expressed in a form which, being equated to zero, represents a bicircular quartic. In fact, if the quartic, being referred to its four Jacobian circles J_1, J_2, J_3, J_4, be expressed in the form

$$\Sigma_4^1 \frac{a_i^2 J_i^2}{k_i^2} = 0,$$

where k_i is the radius of J_i, a_i is the semi-axis major of the corresponding focal conic, and

$$\Sigma_4^1 \frac{J_i^2}{k_i^2} = 0 \text{ identically}$$

(see Casey's *Memoir*), then we can easily find that

$$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$$

is proportional to $\Sigma_4^1 \frac{a_i^4 J_i^2}{k_i^2}$.

Calling this expression V, let us seek the relations connecting the points where a conic having quartic contact with the curve $V+\lambda U$ meets U. We have

$$ds = \sqrt{V}\, du,$$

and

$$\Sigma\, ds = \Sigma \sqrt{V}\, du;$$

hence, from (42), we find

$$\Sigma ds = \Sigma_8^1 \frac{Q}{t\phi'(t)},$$

where $x : y : 1 = (at^2+bt+c) : (a't^2+b't+c') : t$ .....(61),

and Q is the expression of the eighth degree in t, which gives the intersection of the conic with V. We thus have, if

$$Q = a_0 t^8 + \ldots + a_8,$$

Σds proportional to $$a_0 - \frac{a_8}{\varpi},$$

where $$\varpi = t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8.$$

Now, substituting the values (61) in U and V, respectively, we find that ϖ has the same value in each case, and, therefore, that Σds vanishes; that is, a conic having quartic contact with $V+\lambda U$ cuts off from the curve four arcs whose sum is constant. In this result we suppose four arcs taken positively, and the other four negatively.

Again, suppose the quartic

$$V+\lambda U-k^2 = 0$$

to intersect U, then we have

$$\Sigma ds = \Sigma \surd V\, du = k \Sigma du = 0,$$

from which we infer a result similar to the preceding concerning the intercepted arcs.

Considering the central bicircular quartic

$$U \equiv (x^2+y^2+k^2)^2 - 4(a^2x^2+b^2y^2) = 0,$$

we have seen that $$\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$$

for any point on U is proportional to

$$a^2(a^2-k^2)x^2 + b^2(b^2-k^2)y^2 \equiv V.$$

Hence, if the conic $$V - L^2 = 0,$$

where L is a line, meet the curve, we have, for the eight points of intersection, $$\Sigma ds = \Sigma \surd V\, du = \Sigma L\, du = 0.$$

That is, a conic touching the pair of lines represented by $V = 0$ cuts

off from the curve four arcs whose sum is constant. In the case of the ovals of Cassini, $k^2 = a^2 + b^2$, and V is proportional to $x^2 + y^2$, from which we see that for this curve the conic has the origin for one of its foci.

36. With respect to the circular twisted quartic for whose arc I have found an expression above (21), I may notice the envelope of a plane cutting off from the curve two arcs whose sum or difference is rectifiable by circular functions and logarithms. With the notation I have used above, it is easy to see, by a particular case of Abel's theorem, that we must have $\phi + \lambda\psi =$ a perfect square in this case,

where $$\phi = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4,$$

and α, β, γ, δ are the coordinates of the plane. Substituting, then, the roots of ψ successively in this identity, and eliminating the constants in the quadratic expression, we get a result which may be

written $$\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}} + \nu^{\frac{1}{2}} + \rho^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (62),$$

where λ, μ, ν, ρ are the tangential equations of the four points on the curve corresponding to the roots of $\psi = 0$. Now, this equation represents the surface whose equation in tetrahedral coordinates is

$$x^{-1} + y^{-1} + z^{-1} + u^{-1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (63),$$

that is, a quadrinodal cubic. Hence we see that the envelope of the plane is a quadrinodal cubic.

In the case of the particular curve whose arc is expressible by a pure elliptic integral of the first kind, we easily find that the plane cutting off two equal arcs from the curve touches a cone of the fourth class whose equation is of the form

$$l\beta^2\gamma^2 + m\gamma^2\alpha^2 + n\alpha^2\beta^2 = 0.$$

37. For curves of higher orders, it may be noticed that, if the plane curve whose polar equation is

$$r^6 - 2a^3r^3 \cos 3\vartheta + k^6 = 0$$

be met by a circle, we have, for the six points of intersection,

$$\Sigma ds = 0, \quad \text{or} \quad \Sigma s = \text{a constant.}$$

In this case, it may be observed, we cannot take the arcs with different signs.

Third Memoir on Reciprocants. *By* L. J. Rogers, B.A.

[*Read November 11th*, 1886.]

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I.

During the publication of my memoir on Homographic and Circular Reciprocants, I lighted upon a very simple geometrical interpretation of the latter. It was seen (Vol. xvii., p. 231) that the integration of circular reciprocant equations led to complete primitives, which remained unaltered after making certain transformations in the variables. These two transformations were such that any expression of the form $A(x^2+y^2)+Bx+Cy$ assumed the form $\frac{C_1}{C_2}$, where $C_1=0$, $C_2=0$ were equations to circles in the new variables. It must be observed that after the first transformation the numerator of the fraction substituted for y was linear in form, but after the second or correlative transformation the numerator becomes circular.

Hence both x and y, after both transformations, yield expressions of the form $\frac{C_1}{C_0}$, $\frac{C_2}{C_0}$, respectively; $C_1=0$, $C_2=0$ being equations to circles, and $C_0=0$ being a circle of null radius.

This transformation contains 10 constants, and since, in the imaginary form of the transformation, there are only 6, we must evidently have four equations connecting these constants.

Let the transformation be conducted as follows:—

Replace x, y by X, Y, where

$$\left.\begin{aligned} X &= \kappa\,\frac{(x-\alpha)^2+(y-\beta)^2-\rho^2}{(x-\mu)^2+(y-\nu)^2} \\ Y &= \lambda\,\frac{(x-\alpha')^2+(y-\beta')^2-\rho'^2}{(x-\mu)^2+(y-\nu)^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $X+Yi$, expressed in terms of x and y, was seen to be linear both in numerator and denominator. There must be, therefore, a

common factor in numerator and denominator, since at present, by (1), both are of the second degree. The same would be true if we change x into $x+\mu$, and y into $y+\nu$, *i.e.*,

$$\kappa(x+\mu-\alpha)^2+\kappa(y+\nu-\beta)^2-\kappa\rho^2+\lambda i(x+\mu-\alpha')^2+\lambda i(y+\nu-\beta')^2-\lambda i\rho'^2 \quad \ldots\ldots\ldots(2)$$

is divisible by $y+xi$, leaving the factor $x+yi$ in the denominator.

This being true, the absolute term in (2) must vanish. That is,

$$\kappa(\mu-\alpha)^2+\kappa(\nu-\beta)^2-\kappa\rho^2=-\lambda i(\mu-\alpha')^2-\lambda i(\nu-\beta')^2-\lambda i\rho'^2 \quad \ldots(3).$$

Treating $Y+Xi$ the same way, we get the same equation as (2), with every term on the right positive instead of negative.

We get then

$$\left.\begin{aligned}(\mu-\alpha)^2+(\nu-\beta)^2&=\rho^2\\(\mu-\alpha')^2+(\nu-\beta')^2&=\rho'^2\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Moreover, from (2), we also get

$$\kappa(\mu-\alpha)+\iota\lambda(\mu-\alpha')=\iota\kappa(\nu-\beta)-\lambda(\nu-\beta');$$

and from a similar equation, got by treating $Y+Xi$, we get

$$\left.\begin{aligned}\kappa(\mu-\alpha)&=-\lambda(\nu-\beta')\\\lambda(\mu-\alpha')&=\kappa(\nu-\beta)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

making up, in (4) and (5), the requisite number of relations among the constants.

From (5) we deduce

$$(\mu-\alpha)(\mu-\alpha')+(\nu-\beta)(\nu-\beta')=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

From (4) and (5) together we see that

$$\kappa\rho=\lambda\rho' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

By means of these relations we may deduce an expression for X^2+Y^2, which, involving mere algebraical work, it is scarcely worth while to trace out step by step. We have

$$X^2+Y^2=(\kappa^2+\lambda^2)\,\frac{\left(x+\mu-2\frac{\alpha\rho'^2+\alpha'\rho^2}{\rho^2+\rho'^2}\right)^2+\left(y+\nu-2\frac{\beta\rho'^2+\beta'\rho^2}{\rho^2+\rho'^2}\right)^2}{(x-\mu)^2+(y-\nu)^2} \quad \ldots(8),$$

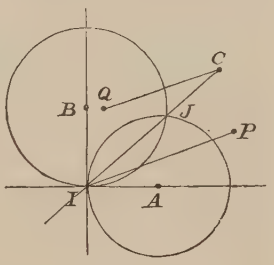
so that X^2+Y^2 comes into the form $\frac{C_4}{C}$, where $C_4=0$ is a circle of

null radius. This can be easily verified from the original imaginary transformation.

We now proceed to interpret these results geometrically.

Let (μ, ν) be the Cartesian coordinates of a point I, A the centre of the circle $(x-\alpha)^2+(y-\beta)^2=\rho^2$, B the centre of $(x-\alpha')^2+(y-\beta')^2=\rho'^2$.

Then, by (4), we see that I is one of the intersections of the two circles. Let J be the other.



By (6), we see that these circles cut orthogonally.

Let (x, y) be the coordinates of any point P, and (X, Y) those of a point Q dependent on P.

It is required to show geometrically how Q may be evolved from P.

For simplicity, let us take IA, IB as coordinate axes. Then we get

$$\left.\begin{aligned} X &= \kappa\frac{x^2+y^2-2\rho x}{x^2+y^2} = \kappa-\frac{2\rho\kappa x}{r^2} \\ Y &= \lambda\frac{x^2+y^2-2\rho' y}{x^2+y^2} = \lambda-\frac{2\rho'\lambda y}{r^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

whence $(X-\kappa)^2+(Y-\lambda)^2=\dfrac{4\rho^2\kappa^2}{r^2}$, because $\rho\kappa=\rho'\lambda$, by (7).

Hence (X, Y) or Q lies on a circle whose centre is (κ, λ), *i.e.*, whose centre lies on the line

$$Y=\frac{\rho}{\rho'}x,$$

that is, on IJ.

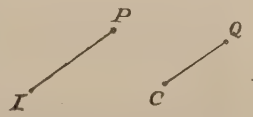
The actual position is arbitrary, dependent on the value chosen for κ.

Let this centre be C.

Again, we see that the radius of the circle on which Q lies is inversely proportional to r, that is, IP.

Further, we see that $\dfrac{Y-\lambda}{X-\kappa}=\dfrac{y}{x}$, so that CQ is parallel to IP.

Clearing the figure, therefore, entirely of all unnecessary parts, we reduce it to the following, and obtain the succeeding method of generating Q.



Let P be any point on a given curve.

Let I, C be any points. From C draw CQ inversely proportional

and parallel to IP; Q will then have been deduced from P by my so-called circular transformation.

It is, in fact, the generalized form of inversion, or rather inversion followed by rectangular transformation. For C may just as well coincide with I at first, and afterwards be transferred to any required point together with the curve generated by Q.

If, then, the curve P lead to a circular reciprocant differential equation, the curve thus generated by Q will be a curve of the same family as P.

Circular reciprocant curves are therefore those which are their own inverses with respect to any point whatever.

II.

We can now proceed to integrate the differential equations formed by equating to zero the two simplest circular reciprocants after $(1+t^2)\,b-3a^2t$, or ϕ_2.

These are $$4\phi_2\phi_4-5\phi_3^2-4\phi_1^2\phi_2^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and $$4\phi_5\phi_2^2-18\phi_4\phi_3\phi_2+4\phi_3\phi_2^2\phi_1^2-8\phi_2^4\phi_1+15\phi_3^3=0 \quad \ldots\ldots\ldots\ldots(2)$$

(see Vol. XVII., p. 230).

The simplest method will be to integrate first the corresponding homographic reciprocant equations, and then transform according to the method given on page 226, Vol. XVII.

The first equation has for its correlative

$$4M_2M_4-5M_3^2+M_1^2M_2^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let us find the differential equation of the homographically generalized form of the equation $y=x^n$.

Here
$$y=x^n,$$
$$t=nx^{n-1},$$
$$M_1=n\,(n-1)\,x^{n-2},$$
$$M_2=-\tfrac{1}{2}n^2\,(n-1)\,(n+1)\,x^{2n-4},$$
$$M_3=\tfrac{1}{2}n^3\,(n-1)\,(n+1)^2\,x^{3n-6},$$
$$M_4=-\tfrac{3}{4}n^4\,(n-1)\,(n+1)^3\,x^{4n-8};$$

whence $$4M_2M_4-5M_3^2+M_1^2M_2^2=\tfrac{1}{2}n^4\,(n^2-1)^2\,(n^2+1)\,x^{6n-12}.$$

We easily see, then, that (3) is satisfied if $n=0$, ± 1, or $\pm\sqrt{-1}$. We discard the real values, since they make M_1, M_2, &c. all vanish

identically, and see that

$$y = x^{\sqrt{-1}}$$

is a solution of (3).

We can easily reduce this to a rational form, for the equation may be written $\log y = i \log x$, or even $\log y = i \log \frac{1-xi}{1+xi}$, since the substitution for x is homographic.

But it is easy to show, by Trigonometry, that

$$i \log \frac{1-xi}{1+xi} = 2 \tan^{-1} x.$$

Hence $\log y = 2 \tan^{-1} x$ is a rational solution of (3), and, by generalizing homographically, we get

$$\kappa + \log \lambda' \frac{y+\mu'}{y+\nu'} = 2 \tan^{-1} \lambda \frac{x+\mu}{x+\nu}.$$

This equation contains seven arbitrary constants, but these may be reduced to the requisite number five by changing κ into $\kappa - \log \lambda' + 2 \tan^{-1} \frac{1-\lambda}{1+\lambda}$, when the equation takes the form

$$\kappa + \log \frac{y+M'}{y+N'} = 2 \tan^{-1} \frac{x+M}{x+N} \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

The general homographic differential equation to $y = x^n$ is

$$(4M_2M_4 - 5M_3^2 + M_1^2M_2^2)(n^2-1) + 4M_2^3(n^2+1) = 0,$$

which is deduced by eliminating x by means of the values found for $4M_2M_4 - 5M_3^2 + M_1^2M^2$ and M_2^3 in the preceding section.

Eliminating n, we get the third homographic protomorph

$$4M_5M_2^2 - 18M_4M_3M_2 + 15M^3 - M^2M_2^2M_3 + 2M_1M_2^4$$

equated to zero as the differential equation of

$$\frac{y+M'}{y+N'} = k \left(\frac{x+M}{x+N}\right)^n \quad \ldots\ldots\ldots\ldots\ldots \quad (6).$$

III.

We can now deduce the solutions of equations (1) and (2) in §II.

Equation (1) gives

$$y + xi = (x+yi)^i \text{ as a primitive,}$$

or
$$i(x^2+y^2) = (x+yi)^{i+1},$$
$$i^i(x^2+y^2)^i = (y+xi)^{i+1};$$

therefore
$$i^{i-1}(x^2+y^2)^{i-1} = \left(\frac{y+xi}{x+yi}\right)^{i+1}.$$

Taking logarithms and using polar coordinates, we get

$$(i-1)\log i + 2(i-1)\log r = (i+1)\log\frac{y+xi}{x+yi}$$
$$= (i+1)\log i\frac{x-yi}{x+yi}$$
$$= (i+1)\log i - 2i(i+1)\tan^{-1}\frac{y}{x};$$

therefore
$$(i-1)\log r = \log i - (i-1)\theta,$$
$$r = i^{i-1}e^{-\theta}.$$

The complete solution of $4\phi_2\phi_4 - 5\phi_3^2 - 4\phi_1^2\phi_2^2 = 0$ is therefore the result of generalizing by inversion the equiangular spiral

$$r = e^{\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This result is easily verified by considering the intrinsic equation to the spiral, $s = e^{n\phi}$.

Here $n\phi_1 = s^{-1}$, $n\phi_2 = -s^{-2}$, $n\phi_3 = 2s^{-3}$, $n\phi_4 = -6s^{-4}$, so that

$$4\phi_2\phi_4 - 5\phi_3^3 - 4\phi_1^2\phi_2^2 = 4s^{-6}\left(\frac{1}{n^2} - \frac{1}{n^4}\right),$$

which vanishes when $n = \pm 1$.

Otherwise, for all other values of n,

$$4\phi_2\phi_4 - 5\phi_3^3 - 4\phi_1^2\phi_2^2 + 4n^3\left(\frac{1}{n^2} - \frac{1}{n^4}\right)\phi_2^3 = 0.$$

Eliminating n, we get equation (2) as the differential equation of the curve obtained by generalizing by inversion any equiangular spiral whatever.

IV.

The chief point of interest in the solution of the differential equations in the foregoing sections is their analogy to the projective differential equations discovered by M. Halphen, and integrated by him in his Thèses (Sur les Invariants Différentiels, No. 408). The

expression there written as Δ is equal, but for a numerical factor, to $6R_3R_5-7R_4^2-3R_3^2R_2$ [see Vol. XVII., p. 349 (8)], and the solution of this expression equated to zero may be obtained as follows :—

Let $y = x^n$,

$t = nx^{n-1}$,

$a = n(n-1)x^{n-2}$, &c.,

so that

$$3R_2 = -n^3(n-1)^2(n-2)(2n-1)x^{2n-6},$$

$$9R_3 = 2n^2(n-1)^2(n-2)(2n-1)(n+1)x^{3n-9},$$

$$9R_4 = -2n^2(n-1)^2(n-2)(2n-1)(n+1)^2x^{4n-12},$$

$$27R_5 = 8n^2(n-1)^2(n-2)(2n-1)(n+1)^3x^{5n-15}.$$

These values of the protomorphs give

$$6R_3R_5-7R_4^2-3R_3^2R_2$$

$$= \frac{4}{27}n^8(n-1)^8(n-2)^2(2n-1)^2(n+1)^2(n^2-n+1)x^{8n-24} \quad \ldots (1).$$

Equating to zero, we get

$$n^3-n+1 = 0,$$

so that the solution of

$$6R_3R_5-7R_4^2-3R_3^2R_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

is $y = x^\omega$,

where ω is one of the imaginary cube roots of -1.

This equation, rationalized as in the preceding sections, gives the equiangular spiral $r = e^{\theta\sqrt{3}}$

(see M. Halphen's *Thèse*, §13).

This curve, generalized by projection, will be the complete primitive of (2).

Eliminating x from (1), and the value given for R_3, we see that the general differential equation for $y = x^n$ is of the form

$$(6R_3R_5-7R_4^2-3R_3^2R_2)^3+kR_3^8 = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

where k is a function of n.

Eliminating k, we get the next projective photomorph equation

$$18R_6R_3^2-72R_5R_4R_3+56R_4^3+6R_4R_3^2R_2-9R_3^4 = 0 \ldots\ldots (4),$$

so that the complete primitive of (3) is the projective generalisation of $y = x^n$.

This equation is written by M. Halphen, $T = 0$.

The equation to cuspidal cubics comes under the head of (3). It will be found to be

$$2^9 . 5^3 . P_8^3 = 3^4 . 7^3 . P_3^8 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

V.

Criterion for distinguishing Projective and Self-polar Reciprocants.

We have already seen that there exist a series of projective reciprocants which are also self-polar; that is, are also reciprocant in r and ρ, where

$$r = \int a^{\frac{1}{3}} dx,$$

and

$$\rho = \int \left(\frac{d^2\eta}{d\xi^2}\right)^{\frac{1}{3}} d\xi,$$

(ξ, η) being Boothian coordinates corresponding to the Cartesian coordinates (x, y).

These projective reciprocants are

$$R_3,$$

$$6R_3R_5 - 7R_4^2 + 3R_3^2R_2, \text{ \&c.,}$$

and are of weight 3, 8, 12, 16, &c.,

and are all positive in character, except R_3, when treated as reciprocants in x and y.

But the same is not true when they are looked upon as reciprocants in r and ρ.

Let us call these projective protomorphs

$$P_3,\ 6P_8,\ P_{12}, \ldots \text{ respectively,}$$

so that any isobaric function of the P's is a projective reciprocant. The negative character of P_3 will never affect the result; for it must always enter as a fourth power, since all the suffixes are multiples of 4 except this first.

Treated as reciprocants in r and ρ,

P_3 is negative,

P_8 is positive,

P_{12} is negative, &c.,

the sign alternating.

Hence we easily see that, if

$$F(P_3, P_8, P_{12}, P_{16}, P_{20}, \ldots) = 0$$

be the reciprocant differential equation of a projective curve, then its polar reciprocal will be

$$F(-P_3, P_8, -P_{12}, P_{16}, -P_{20}, \ldots) = 0,$$

and, if the same equation be obtained, the curve will also be self-polar.

As an example of a self-polar curve, we may cite the curve

$$y = x^n,$$

which yields, when projectively generalized, the equation

$$P_8^3 + kP_3^8 = 0$$

[see § IV., (3)], where k is a function of n; or, eliminating n,

$$P_{12} = 0$$

[see § IV., (4)], which equations both satisfy the self-polar criterion.

Hence $y = x^n$ is a self-polar curve, as is easily proved by finding its tangential equation. All cuspidal cubics come under this heading, n being equal to 3.

This criterion for distinguishing between projective and self-polar reciprocants is virtually given by M. Halphen in the above-mentioned *Thèse*, § 39, &c.

VI.

I conclude by remarking on a curious property of self-polar reciprocants, viz., that any self-polar reciprocant is also a reciprocant in x and t.

In other words, if we change $a, b, c \ldots$ into $t, a, b \ldots$ in any projective, we shall get a mixed homogeneous reciprocant.

The truth of this statement is apparent in the two best known cases, viz., a and $9a^2d - 45abc + 40b^3$, which become t and

$$9t(tc - 5ab) + 40a^3,$$

the latter being the sum of $9t$ times the post-Schwarzian and a^3, both negative reciprocants of characteristic 9.

Let us now attempt to express

$$6R_3R_5-7R_4^2-3R_3^2R_2,$$

after the similar transformation in terms of the M-protomorphs. If the M-function is of even degree in every term, or of odd degree in every term, then we shall get a reciprocant.

Let $$F_2 = tb-\tfrac{5}{3}a^3,$$

$$F_3 = t^2c-5tab+\tfrac{40}{9}a^3, \qquad \&c.;$$

in short, let R_m become F_m when $a, b \ldots$ are changed into $t, a, b \ldots$

Then
$$F_{m+1} = \left(t\delta_x-\frac{4m}{3}a\right)F_m$$

$$= \left(t\delta_x-\frac{3m}{2}a\right)F_m+\frac{m}{6}aF_m$$

$$= G_mF_m+\frac{m}{6}aF_m.$$

Thus,
$$F_2 = tb-\tfrac{3}{2}a^3-\tfrac{1}{6}a^2 = M_2-\tfrac{1}{6}M_1^2,$$
$$F_3 = G_m(M_2-\tfrac{1}{6}M_1^2)+\tfrac{1}{3}aF_2$$
$$= M_3-\tfrac{1}{3}M_1M_2+\tfrac{1}{3}M_1M_2-\tfrac{1}{18}M_1^3$$
$$= M_3-\tfrac{1}{18}M_1^3,$$

which is a reciprocant.

Furthermore,
$$F_4 = G_m(M_3-\tfrac{1}{18}M_1^3)+\tfrac{1}{2}aF_3$$
$$= M_4-\tfrac{1}{6}M_1^2M_2+\tfrac{1}{2}M_1M_3-\tfrac{1}{36}M_1^4,$$
$$F_5 = G_mF_4+\tfrac{2}{3}aF_4$$
$$= M_5-\tfrac{1}{3}M_1M_2^2+\tfrac{1}{6}M_1^2M_3+\tfrac{1}{2}M_2M_3+\tfrac{7}{6}M_1M_4-\tfrac{2}{9}M_1^3M_2-\tfrac{1}{54}M_1^5.$$

Given these values, we have to see whether

$$6F_3F_5-7F_4^2-3F_3^2F_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

be a reciprocant. Substituting, we get

$$6(M_3-\tfrac{1}{18}M_1^3)(M_5-\tfrac{1}{3}M_1M_2^2+\tfrac{1}{6}M_1^2M_3+\tfrac{1}{2}M_2M_3+\tfrac{7}{6}M_1M_4-\tfrac{2}{9}M_1^3M_2-\tfrac{1}{54}M_1^5)$$
$$-7(M_4+\tfrac{1}{2}M_1M_3-\tfrac{1}{6}M_1^2M_2-\tfrac{1}{36}M_1^4)^2$$
$$-3(M_2-\tfrac{1}{6}M_1^2)(M_3-\tfrac{1}{18}M_1^3)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now, let us portion off the terms of uneven degree. We get

$$\begin{aligned} &6\,(M_3-\tfrac{1}{18}M_1^3)(\tfrac{1}{2}M_2M_3+\tfrac{7}{6}M_1M_4-\tfrac{2}{9}M_1^3M_2)\\ &-14\,(M_4-\tfrac{1}{6}M_1^2M_2)(\tfrac{1}{2}M_1M_3-\tfrac{1}{36}M_1^4)\\ &-3M_2\,(M_3-\tfrac{1}{18}M_1^3)^2. \end{aligned}$$

Omitting the factor $M_3-\frac{1}{18}M_1^3$ running through, we get

$$6\,(\tfrac{1}{2}M_2M_3+\tfrac{7}{6}M_1M_4-\tfrac{2}{9}M_1^3M_2)-7M_1\,(M_4-\tfrac{1}{6}M_1^2M_2)-3M_2\,(M_3-\tfrac{1}{18}M_1^3),$$

which is easily seen to vanish identically. Hence (2) only contains terms of even degree, and we see thereby that

$$6F_3F_5-7F_4^2-3F_3^2F_2$$

is a reciprocant (positive) of degree 8 and characteristic 24.

Now, from R_3 and $6R_3R_5-7R_4^2-3R_3^2R_2$, of weights 3 and 8, deduce another projective reciprocant of weight 12, by § 3 (2).

It is obvious, then, that from F_3 and $6F_3F_5-7F_4^2-3F_3^2F_2$, whose characteristics are in the ratio of 3 to 8, we shall deduce an F-function of exactly the same form as the R-function just obtained.

The one only differs by a power of a from

$$3R_3\delta_x(6R_3R_5-7R_4^2-3R_3^2R_2)-8\,(6R_3R_5-7R_4^2-3R_3^2R_2)\,\delta_xR_3,$$

while the other is the same, with F instead of R, except for some power of t as a through factor. We get, then,

$$R_6R_3^2+\ldots \text{ from one pair,}$$

and

$$F_6F_3^2+\ldots \text{ from the other,}$$

since, instead of x, we may differentiate for a function f, given by the equation

$$\frac{df}{dx}=t^{\frac{1}{2}},$$

so that

$$\frac{d}{dx}=\frac{1}{t^{\frac{1}{2}}}\cdot\frac{d}{df}.$$

The weight of $F_6F_3^2+\ldots$ bears the same ratio to that of F_3 as that of $R_6R_3^2+\ldots$ does to that of R_3. Hence with the latter two we deduce a new R-function, and from the former two we deduce the same F-function, just as in the last paragraph. We get then a series of F-functions, all reciprocants, of exactly the same form as the

similarly deduced series of R-functions. But these latter are the projective protomorphs from which all others can be formed. Hence from any projective R-function we can deduce an F-function which will be a reciprocant. For in both R-function and F-function the character is $(-1)^\omega$, and if the R-protomorphs be combined in any way so that every term is of the same character, so also will be the case with the same function of the F-protomorphs.

Hence the truth of the statement at the beginning of this section is established.

December 9th, 1886.

SIR JAMES COCKLE, F.R.S., President, in the Chair.

Mr. F. S. Macaulay was admitted into the Society.

Prof. Dairoku Y. Kikuchi, B.A. Cambridge, Professor of Mathematics at Tokio, was elected a member.

The following communications were made:—

The Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants: E. B. Elliott.

Circular Notes: R. Tucker.

The Problem of the Duration of Play: Capt. MacMahon.

Note on two Annihilators in the Theory of Elliptic Functions: J. Griffiths.

Mr. Hammond made some remarks on the communication read by Capt. MacMahon at the November meeting.

The following presents were made:—

"Physical Society of London—Proceedings," Vol. VIII., Part II.; Oct., 1886.

"Educational Times," for December.

"Johns Hopkins University Circulars," Vol. VI., No. 52; November, 1886.

"Proceedings of the Canadian Institute, Toronto," Vol. IV., Fasc. 1; Toronto, 1886.

"Bulletin des Sciences Mathématiques," Nov., 1886.

"Acta Mathematica," 9, 1.

"Journal für Mathematik," Crelle, Band C., Heft 3.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 8; Roma, 1886.

"Bollettino delle Pubblicazioni Italiane recevute per Diritto di Stampa," Nos. 21 and 22; Firenze, 1886.

"Über eine Aufgabe der Ausgleichungsrechnung," von H. Bruns; 4to pamphlet; Leipzig, 1886.

"Zur Geschichte des menschlichen Rückenmarkes und der Nervenwurzeln," von Wilhelm His; 4to pamphlet; Leipzig, 1886.

Nine pamphlets (4to) by L. Kronecker: from the author.

On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants. *By* E. B. Elliott.

[*Read Dec. 9th*, 1886.]

Contents.

§ 1. Introduction.
§ 2. Illustration of Method by its Application to Ordinary Reciprocants.
§ 3. Notation and First Principles.
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§§ 7, 8. Second, or z to y Transformation.
§ 9. Deduction of Laws obeyed by Pure Ternary Reciprocants.
§ 10. These Laws Sufficient as well as Necessary.
§ 11. The two first Pure Ternary Reciprocants.
§ 12. Determination of a third, and indication of means of educing an infinity of others.

1. Mr. Leudesdorf (*Proceedings*, Vol. xvii., pp. 216—219) has noticed that a pure ternary reciprocant must have two quadro-linear differential annihilators, and has calculated a few terms of one of them. In the present paper a systematic search is made for a complete system of annihilators; and the results may, it will be found, be summarised as follows. Every pure ternary reciprocant, regarded as a function of the derivatives of a dependent variable z with regard to two independent variables x and y, must satisfy six linear partial differential equations, forming three pairs; and conversely, a homogeneous function of the second and higher derivatives, which does satisfy the six equations, must necessarily be a reciprocant. Two of the six equations are of forms akin to Euler's equation of homogeneous functions, involve a single constant which is different for

different reciprocants, and express facts with regard to degree and weight, homogeneity and isobarism. Two more are lineo-linear, and express that any pure ternary reciprocant is a full invariant of the system of emanants, as was proved in my former paper on ternary reciprocants (Vol. XVII., p. 181). The remaining pair are quadro-linear, that is to say, linear in the differential coefficients, but with coefficients quadratic in the elements, and are directly analogous to the well-known equation $VR = 0$ of ordinary or binary reciprocants, which equation either of them includes.

Other results which it is hoped may prove to have interest will be arrived at incidentally.

2. In this article the method to be followed will be exemplified by its application to the known theory of ordinary or binary reciprocants. This course seems justifiable, as the process has some novelty of form, and as the annihilator V will be arrived at in a compact symbolical shape, not, I believe, hitherto noticed.

Let, for every positive integral value of r, y_r denote $\frac{1}{r!}\frac{d^r y}{dx^r}$, so that $y_1, y_2, y_3, y_4, \ldots$ are the $t, a, b, c, \ldots$ of Professor Sylvester, and the $y_1, \frac{y_2}{2!}, \frac{y_3}{3!}, \frac{y_4}{4!}, \ldots$ of Mr. Leudesdorf. Then, if ξ, η be corresponding finite increments of x and y, Taylor's theorem tells us that

$$y_2\xi^2+y_3\xi^3+y_4\xi^4+\ldots = \eta-y_1\xi = -y_1(\xi-x_1\eta)$$

$$= -\frac{1}{x_1}(x_2\eta^2+x_3\eta^3+x_4\eta^4+\ldots) \quad \ldots\ldots(1),$$

where x_r denotes $\frac{1}{r!}\frac{d^r x}{dy^r}$.

Now, suppose the form of the connecting relation between x and y so to change that x_1 alters infinitesimally, but $x_2, x_3, x_4, \ldots$ remain constant. Also let the increment η remain unaltered. The corresponding ξ for which (1) remains true will, in general, receive a change. By differentiation of (1), therefore,

$$\frac{dy_2}{dx_1}\xi^2+\frac{dy_3}{dx_1}\xi^3+\frac{dy_4}{dx_1}\xi^4+\ldots+(2y_2\xi+3y_3\xi^2+4y_4\xi^3+\ldots)\frac{d\xi}{dx_1}$$

$$= \frac{1}{x_1^2}(x_2\eta^2+x_3\eta^3+x_4\eta^4+\ldots)$$

$$= -\frac{1}{x_1}(y_2\xi^2+y_3\xi^3+y_4\xi^4+\ldots), \text{ by (1).}$$

Moreover, since $\xi = x_1\eta + x_2\eta^2 + x_3\eta^3 + \dots,$

$$\frac{d\xi}{dx_1} = \eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots.$$

Hence we obtain, upon substitution,

$$\frac{dy_2}{dx_1}\xi^2 + \frac{dy_3}{dx_1}\xi^3 + \frac{dy_4}{dx_1}\xi^4 + \dots$$

$$= -(y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots)(2y_2\xi + 3y_3\xi^2 + 4y_4\xi^3 + \dots)$$

$$-y_1(y_2\xi^2 + y_3\xi^3 + y_4\xi^4 + \dots) \quad \dots\dots\dots\dots\dots\dots(2);$$

so that $\frac{dy_2}{dx_1}$, $\frac{dy_3}{dx_1}$, $\frac{dy_4}{dx_1}$, ... are the coefficients of ξ^2, ξ^3, ξ^4, ... in the expanded right-hand side of this identity. In other words,

$$\frac{dy_2}{dx_1} = -3y_1y_2,$$

$$\frac{dy_3}{dx_1} = -4(y_1y_3 + \tfrac{1}{2}y_2^2),$$

$$\frac{dy_4}{dx_1} = -5(y_1y_4 + y_2y_3),$$

$$\frac{dy_5}{dx_1} = -6(y_1y_5 + y_2y_4 + \tfrac{1}{2}y_3^2),$$

$$\dots\dots\dots,$$

so that, since also $\frac{dy_1}{dx_1} = -y_1^2$, if ϕ is any function whatever,

$$-\frac{d}{dx_1}\phi(y_1, y_2, y_3, y_4, \dots)$$

$$= y_1^2\frac{d\phi}{dy_1} + 3y_1y_2\frac{d\phi}{dy_2} + 4(y_1y_3 + \tfrac{1}{2}y_2^2)\frac{d\phi}{dy_3}$$

$$+5(y_1y_4 + y_2y_3)\frac{d\phi}{dy_4} + 6(y_1y_5 + y_2y_4 + \tfrac{1}{2}y_3^2)\frac{d\phi}{dy_5}$$

$$+7(y_1y_6 + y_2y_5 + y_3y_4)\frac{d\phi}{dy_6} + \dots$$

$$= y_1\left(y_1\frac{d\phi}{dy_1} + 3y_2\frac{d\phi}{dy_2} + 4y_3\frac{d\phi}{dy_3} + \dots\right) + V\phi \quad \dots\dots\dots\dots\dots(3),$$

where V denotes, as usual,

$$4\frac{y_2^2}{2}\frac{d}{dy_3}+5y_2y_3\frac{d}{dy_4}+6\,(y_2y_4+\tfrac{1}{2}y_3^2)\frac{d}{dy_5}+\ldots.$$

This is, if we remember the difference of notation, Mr. Leudesdorf's formula [*Proceedings*, Vol. XVII., p. 199, (1)]; so that the rest of the argument may proceed as in his paper.

Now, the terms free from y_1 in (2) are those which give the coefficients that occur in V. But these terms are

$$-(y_2\xi^2+y_3\xi^3+y_4\xi^4+\ldots)(2y_2\xi+3y_3\xi^2+4y_4\xi^3+\ldots),$$

which may be written

$$-\frac{1}{2}\frac{d}{d\xi}\{(y_2\xi^2+y_3\xi^3+y_4\xi^4+\ldots)^2\}.$$

We have, then, that the coefficients of $\frac{d}{dy_3}$, $\frac{d}{dy_4}$, $\frac{d}{dy_5}$, ... in V are exactly the coefficients with sign changed of ξ^3, ξ^4, ξ^5, ... in this expansion. The expression with sign changed may consequently be itself taken as a symbolical representation of V, becoming V as it does when, after the expansion and differentiation are performed, for every power ξ^r in it the corresponding $\frac{d}{dy_r}$ is substituted. Or we may say, still more briefly, that, symbolically,

$$2V=\frac{d}{d\xi}\{(\eta-y_1\xi)^2\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

the meaning being that, in this right-hand side, η, regarded as an increment of y, is to be expanded by Taylor's theorem in terms of powers of ξ, that the squaring and partial differentiation with regard to ξ are to be performed, and that then, for all values of the number r, ξ^r is to be replaced by the symbol of partial differentiation $\frac{d}{dy_r}$.

Other results can readily be obtained by differentiation of (1) partially with regard to each of the derivatives x_r. The method is exactly that of the case already discussed of differentiation with regard to x_1. The process is, however, even more simple for values of r exceeding unity, and need not be here given, as the analogous reasoning when there are three variables x, y, z will be gone through at length in § 6 below. The result is that, r being any number but unity, and the expression operated on upon the right being any function whatever of y_1, y_2, y_3, ..., whose unknown equivalent in terms of

$x_1, x_2, x_3, \ldots$ is referred to on the left,

$$(r+1)\frac{d}{dx_r} = -\frac{d}{d\xi}\left\{(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^{r+1}\right\}$$

$$= -\frac{d}{d\xi}(\eta^{r+1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

the symbolization on the right being exactly as before.

It may be noticed, in passing, that the terms in the dexter of (5) which do not involve y_1, *i.e.*, the terms symbolically represented by

$$\frac{d}{d\xi}\left\{(\eta - y_1\xi)^{r+1}\right\},$$

constitute Captain MacMahon's operator

$$(2r+2,\ 1\ ;\ r+1,\ 2r-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5a).$$

Hence it follows, without difficulty, that this operator is an annihilator of *pure* functions where transforms do not involve x_r.*

3. Passing now to our main subject, the corresponding theory when there are three variables x, y, z connected by a single relation, let ξ, η, ζ be corresponding finite increments of those variables. Two of these may be chosen at will, and the third is then determinate in virtue of the relation connecting x, y, z. The notation used in my former paper must be discarded as unsuitable for general investigations, and as unfortunate in its peculiar choice of suffixes. The most suitable for adoption would seem to be the following. Let, for all possible combinations of two numbers r, s, $\dfrac{1}{r!\,s!}\dfrac{d^{r+s}z}{dx^r dy^s}$ be represented

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* Note that the most general MacMahon operator of four elements $(\mu, \nu\ ;\ m, n)$ may be written symbolically,

$$\nu\frac{d}{d\xi}\left\{\xi^{\mu/\nu - 2m}(\eta - y_1\xi)^m\right\},$$

in which, ξ and η being regarded in the first place as corresponding increments of x and y, the substitution for η in terms of ξ by Taylor's theorem, the expansion by the multinomial theorem, and the partial differentiation with regard to ξ, are to be performed, and then, for each value of s occurring, $\xi^{s+\mu/\nu}$ is to be replaced by the operative symbol $\dfrac{d}{dy_{n+s+3}}$, which in the Sylvester-MacMahon notation is called $\dfrac{d}{da_{n+s+1}}$. This remark may probably lead to the interpretation of other large classes of these operators, as well as of those of (5*a*) above.

The close connection may be noticed between this symbolization and that of Mr. Hammond on p. 63 of the present volume. The two were simultaneously and independently arrived at, and were communicated to the Society on the same evening.

by z_{rs}, $\frac{1}{r!\,s!}\frac{d^{r+s}x}{dy^r dz^s}$ by x_{rs}, and $\frac{1}{r!\,s!}\frac{d^{r+s}y}{dz^r dx^s}$ by y_{rs}. The connection between ξ, η, ζ may then be written, by Taylor's theorem, in three forms:

$$\left.\begin{aligned}\zeta - z_{10}\xi - z_{01}\eta &= (z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) + (z_{30}\xi^3 + z_{21}\xi^2\eta + z_{12}\xi\eta^2 + z_{03}\eta^3) + \dots \\ \xi - x_{10}\eta - x_{01}\zeta &= (x_{20}\eta^2 + x_{11}\eta\zeta + x_{02}\zeta^2) + (x_{30}\eta^3 + x_{21}\eta^2\zeta + x_{12}\eta\zeta^2 + x_{03}\zeta^3) + \dots \\ \eta - y_{10}\zeta - y_{01}\xi &= (y_{20}\zeta^2 + y_{11}\zeta\xi + y_{02}\xi^2) + (y_{30}\zeta^3 + y_{21}\zeta^2\xi + y_{12}\zeta\xi^2 + y_{03}\xi^3) + \dots\end{aligned}\right\} \quad \dots\dots\dots\dots(6);$$

of which the second [*Proceedings*, Vol. XVII., p. 182 (32)] is only the first divided by either of the equal quantities

$$-z_{10},\ -\frac{1}{x_{01}},\ \left(\frac{z_{10}z_{01}}{x_{10}x_{01}}\right)^{\frac{1}{2}},$$

and the third is the first divided by either

$$-z_{01},\ -\frac{1}{y_{10}},\ \text{or}\ \left(\frac{z_{10}z_{01}}{y_{10}y_{01}}\right)^{\frac{1}{2}}.$$

We have, for instance, from the first two, the equivalence

$$(z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) + (z_{30}\xi^3 + z_{21}\xi^2\eta + z_{12}\xi\eta^2 + z_{03}\eta^3) + \dots$$

$$= -\frac{1}{x_{01}}\left\{(x_{20}\eta^2 + x_{11}\eta\zeta + x_{02}\zeta^2) + (x_{30}\eta^3 + x_{21}\eta^2\zeta + x_{12}\eta\zeta^2 + x_{03}\zeta^3) + \dots\right\} \quad \dots\dots\dots\dots(7).$$

This will now help us to express the partial differential coefficients of functions of the derivatives z_{rs} of z with regard to the derivatives x_{mn} of x, of which they can of course be regarded as functions. Primarily we seek the effect of the operations $\frac{d}{dx_{10}}$, $\frac{d}{dx_{01}}$, where x_{10}, x_{01} are the first derivatives $\frac{dx}{dy}$, $\frac{dx}{dz}$ of x, upon z-functions.

4. First let the connecting relation between x, y, z be so varied infinitesimally, that x_{01} receives an infinitesimal increment, while x_{10}, x_{20}, x_{11}, x_{02}, x_{30}, ... remain unaltered. Keep also the η and ζ in (6) and (7) unchanged. The corresponding ξ for which they remain true will vary infinitesimally in consequence of the change in x_{01}.

Thus, from (7),

$$\left(\frac{dz_{20}}{dx_{01}}\xi^2+\frac{dz_{11}}{dx_{01}}\xi\eta+\frac{dz_{02}}{dx_{01}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{01}}\xi^3+\frac{dz_{21}}{dx_{01}}\xi^2\eta+\dots\right)+\dots$$

$$+\frac{d\xi}{dx_{01}}\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\}$$

$$=\frac{1}{x_{01}^2}\left\{(x_{20}\eta^2+x_{11}\eta\zeta+x_{02}\zeta^2)+(x_{30}\eta^3+x_{21}\eta^2\zeta+x_{12}\eta\zeta^2+x_{03}\zeta^3)+\dots\right\}$$

$$=-\frac{1}{x_{01}}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots\right\},$$

by (7). But, by the second of equations (6),

$$\frac{d\xi}{dx_{01}}=\zeta$$

$$=(z_{10}\xi+z_{01}\eta)+(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\dots.$$

Consequently, upon inserting this value above,

$$\left(\frac{dz_{20}}{dx_{01}}\xi^2+\frac{dz_{11}}{dx_{01}}\xi\eta+\frac{dz_{02}}{dx_{01}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{01}}\xi^3+\dots\right)+\left(\frac{dz_{40}}{dx_{01}}\xi^4+\dots\right)+\dots$$

$$=-\left\{(z_{10}\xi+z_{01}\eta)+(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+\dots\right\}$$

$$\times\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}$$

$$-z_{10}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}$$

$$=-z_{10}\left(\xi\frac{d}{d\xi}+1\right)\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}$$

$$-z_{01}\eta\frac{d}{d\xi}\quad\{\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\}$$

$$-\frac{1}{2}\frac{d}{d\xi}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+\dots)+\dots\right\}^2\right]\quad\dots\dots\dots(8).$$

Thus $\frac{dz_{20}}{dx_{01}}$, $\frac{dz_{11}}{dx_{01}}$, $\frac{dz_{02}}{dx_{01}}$, $\frac{dz_{30}}{dx_{01}}$, ... are the coefficients of ξ^2, $\xi\eta$, η^2, ξ^3, ... in the expansion after differentiation for ξ of this right-hand side. Also, since [*Proceedings*, Vol. XVII., p. 174, (4)],

$$z_{10}=\frac{1}{x_{01}},\quad\text{and}\quad z_{01}=-\frac{x_{10}}{x_{01}},$$

we have, further,

$$\frac{dz_{10}}{dx_{01}} = -\frac{1}{x_{01}^2} = -z_{10}^2, \text{ and } \frac{dz_{01}}{dx_{01}} = \frac{x_{10}}{x_{01}^2} = -z_{10}z_{01} \ldots\ldots\ldots (8a).$$

The means have been obtained, therefore, of writing down the partial differential coefficient with regard to x_{01} of the function of the suffixed x's which is equal to any given function of the suffixed z's. Thus, if

$$\mathfrak{F}(z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, z_{30}, \ldots) = F(x_{10}, x_{01}, x_{20}, x_{11}, x_{02}, x_{30}, \ldots) \ldots (9),$$

then $$\frac{dF}{dx_{01}} = \frac{d\mathfrak{F}}{dz_{10}} \cdot \frac{dz_{10}}{dx_{01}} + \frac{d\mathfrak{F}}{dz_{01}} \cdot \frac{dz_{01}}{dx_{01}} + \frac{d\mathfrak{F}}{dz_{20}} \cdot \frac{dz_{20}}{dx_{01}} + \frac{d\mathfrak{F}}{dz_{11}} \cdot \frac{dz_{11}}{dx_{01}} + \ldots$$

$$= -z_{10}\left\{ z_{10}\frac{d\mathfrak{F}}{dz_{10}} + z_{01}\frac{d\mathfrak{F}}{dz_{01}} \right\}$$

+ the result of writing in the right-hand member of (8), after differentiation and expansion, $\frac{d\mathfrak{F}}{dz_{rs}}$ instead of $\xi^r\eta^s$ for each pair of values of r and s.

If, then, we adopt $\xi^r\eta^s$, in an expanded form, as a symbolical representation of $\frac{d}{dz_{rs}}$,

$$\frac{d}{dx_{01}} = -z_{10}\{z_{10}\xi + z_{01}\eta + (3z_{20}\xi^2 + 2z_{11}\xi\eta + z_{02}\eta^2) + (4z_{30}\xi^3 + 3z_{21}\xi^2\eta + 2z_{12}\xi\eta^2 + z_{03}\eta^3) + \ldots\}$$

$$-z_{01}\{ (2z_{20}\xi\eta + z_{11}\eta^2) + (3z_{30}\xi^2\eta + 2z_{21}\xi\eta^2 + z_{12}\eta^3) + \ldots\}$$

$$-\frac{1}{2}\frac{d}{d\xi}\Big[\{(z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) + (z_{30}\xi^3 + z_{21}\xi^2\eta + z_{12}\xi\eta^2 + z_{03}\eta^3) + \ldots\}^2\Big] \ldots\ldots (10).$$

Now, suppose the function $\mathfrak{F}(z_{10}, \ldots)$, in the identity (9), to be such that its transform F does not involve x_{01}. The right-hand side of (10) will then be an annihilator of $\mathfrak{F}(z_{10}, \ldots)$, and, conversely, for this to be an annihilator will be a sufficient condition for the absence of x_{01}, no less than a necessary consequence of such absence.

To particularise, take for $\mathfrak{F}(z_{10}, \ldots)$ an expression of the form $z_{10}^{-\mu}P$, where P is a pure function of the suffixed z's, *i.e.*, one which does not

involve the first derivatives z_{10}, z_{01}. It results that

$$\frac{d}{dx_{01}}\left(\frac{P}{z_{10}^{\mu}}\right)=\frac{1}{z_{10}^{\mu-1}}\left\{\mu-\left(3z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+z_{02}\frac{d}{dz_{02}}\right)\right.$$

$$\left.-\left(4z_{30}\frac{d}{dz_{30}}+3z_{21}\frac{d}{dz_{21}}+2z_{12}\frac{d}{dz_{12}}+z_{03}\frac{d}{dz_{03}}\right)-\ldots\right\}P$$

$$-\frac{z_{01}}{z_{10}^{\mu}}\left\{\left(2z_{20}\frac{d}{dz_{11}}+z_{11}\frac{d}{dz_{02}}\right)+\left(3z_{30}\frac{d}{dz_{21}}+2z_{21}\frac{d}{dz_{12}}+z_{12}\frac{d}{dz_{03}}\right)+\ldots\right\}P$$

$$-\frac{1}{z_{10}^{\mu}}\cdot\frac{1}{2}\frac{d}{d\xi}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]P,$$

.....(11),

in which the last line is symbolical; and consequently that, if the transform of $z_{10}^{-\mu}P$ is independent of x_{01}, P must satisfy the linear partial differential equation obtained by equating the right-hand member of this identity to zero; and conversely.

But we can say more than this. The right-hand member of (11) consists of three distinct parts, having factors $\frac{1}{z_{10}^{\mu-1}}$, $\frac{z_{01}}{z_{10}^{\mu}}$, $\frac{1}{z_{10}^{\mu}}$, respectively, and not containing either z_{10} or z_{01} in their other factors. Now no term whose z_{10} factor is $z_{10}^{-\mu+1}$ can disappear when added to another term whose z_{10} factor is $z_{10}^{-\mu}$. Nor can a term having z_{01} as a factor be cancelled by addition to a term not involving z_{01}. If, then, the right-hand side of (11) vanish, its three parts must vanish separately. We conclude that, for the transform of $z_{10}^{-\mu}P$ not to involve x_{01}, the necessary and sufficient conditions are three in number; viz., that P satisfy the three linear partial differential equations $E_1P=0$, $\Omega_1P=0$, $V_1P=0$, where the meanings of the three annihilators are

$$E_1\equiv\mu-\left(3z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+z_{02}\frac{d}{dz_{02}}\right)$$

$$-\left(4z_{30}\frac{d}{dz_{30}}+3z_{21}\frac{d}{dz_{21}}+2z_{12}\frac{d}{dz_{12}}+z_{03}\frac{d}{dz_{03}}\right)-\ldots \quad .....(A),$$

$$\Omega_1\equiv \left(2z_{20}\frac{d}{dz_{11}}+z_{11}\frac{d}{dz_{02}}\right)+\left(3z_{30}\frac{d}{dz_{21}}+2z_{21}\frac{d}{dz_{12}}+z_{12}\frac{d}{dz_{03}}\right)+\ldots \quad ...(B),$$

$$V_1\equiv \frac{1}{2}\frac{d}{d\xi}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]$$

$$\equiv \frac{1}{2}\frac{d}{d\xi}\left\{(\zeta-z_{10}\xi-z_{01}\eta)^2\right\} \ldots\ldots\ldots\

the last of which is symbolically written, and is to be taken in the sense that ζ, an increment of z, is first to be expanded in terms of ξ, η, corresponding increments of x and y, by Taylor's theorem, that the expansion and the partial differentiation with regard to ξ are to be performed, and that then, in the result, $\xi^r\eta^s$ is to be replaced by $\frac{d}{dz_{rs}}$ for every pair of values of r and s.

The first two of the three conditions are readily interpreted. Thus, $E_1P = 0$ tells us that, if P be homogeneous, as we shall see later must be the case when it is a reciprocant, it is necessarily isobaric in first suffixes, *i.e.*, such that in every term of which it consists the sum of the first suffixes of the factors is constant; and also that, if w_1 be this constant sum, the first partial weight say, and i the degree of P, then

$$i + w_1 = \mu.$$

Again, $\Omega_1 P = 0$ expresses that P is a ξ-seminvariant, or at any rate a sum of such seminvariants, of the system of emanants

$$(z_{20}, z_{11}, z_{02} \mathbin{\text{\textbigtimes}} \xi, \eta)^2, \quad (z_{30}, z_{21}, z_{12}, z_{03} \mathbin{\text{\textbigtimes}} \xi, \eta)^3, \text{ \&c.};$$

in connection with which see my former paper on "Ternary Reciprocants" (Vol. XVII., pp. 181, 182).

The third condition $V_1P = 0$, a quadro-linear equation, is the first analogue of the equation $VR = 0$ [see (3) or (4) above] of ordinary pure reciprocants.

5. The determination of an expression for $\frac{d}{dx_{10}}$ in terms of the operators $\frac{d}{dz_{rs}}$ is a much shorter matter than that of $\frac{d}{dx_{01}}$. Returning to (7), let x_{10} vary, while η, ζ and all the doubly suffixed x's except x_{10} remain constant. We thus obtain

$$\left(\frac{dz_{20}}{dx_{10}}\xi^2 + \frac{dz_{11}}{dx_{10}}\xi\eta + \frac{dz_{02}}{dx_{10}}\eta^2\right) + \left(\frac{dz_{30}}{dx_{10}}\xi^3 + \frac{dz_{21}}{dx_{10}}\xi^2\eta + \frac{dz_{12}}{dx_{10}}\xi\eta^2 + \frac{dz_{03}}{dx_{10}}\eta^3\right) + \ldots$$

$$= -\frac{d\xi}{dx_{10}} \cdot \frac{d}{d\xi}\left\{(z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) + (z_{30}\xi^3 + z_{21}\xi^2\eta + z_{12}\xi\eta^2 + z_{03}\eta^3) + \ldots\right\};$$

and in this, since by the second of equations (6) $\frac{d\xi}{dx_{10}} = \eta$, the right-hand member may be written

$$-\left\{(2z_{20}\xi\eta + z_{11}\eta^2) + (3z_{30}\xi^2\eta + 2z_{21}\xi\eta^2 + z_{12}\eta^3) + \ldots\right\};$$

whence, equating coefficients,

$$\frac{dz_{20}}{dx_{10}}=0,\quad \frac{dz_{11}}{dx_{10}}=-2z_{20},\quad \frac{dz_{02}}{dx_{10}}=-z_{11},$$

$$\frac{dz_{30}}{dx_{10}}=0,\quad \frac{dz_{21}}{dx_{10}}=-3z_{30},\quad \frac{dz_{12}}{dx_{10}}=-2z_{21},\quad \frac{dz_{03}}{dx_{10}}=-z_{12},$$

&c., &c.,

which, with the facts obtained as in (8*a*),

$$\frac{dz_{10}}{dx_{10}}=0 \text{ and } \frac{dz_{01}}{dx_{10}}=-\frac{1}{x_{01}}=-z_{10},$$

give the equivalence $$\frac{d}{dx_{10}}=-z_{10}\frac{d}{dz_{01}}-\Omega_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where Ω_1 denotes the lineo-linear operator of result (B).

In particular, P denoting a pure function of the doubly suffixed z's, as in the last article,

$$\frac{d}{dx_{10}}\left(\frac{P}{z_{10}^{\mu}}\right)=-\frac{1}{z_{10}^{\mu}}\Omega_1 P \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

which gives us no new information as to such pure functions P as produce in this manner pure functions of the suffixed x's, since the condition $\Omega_1 P=0$ is one of those previously obtained, but tells us that x_{10} cannot possibly occur in the transform of $z_{10}^{-\mu}P$, where P is pure, without x_{01} occurring also.

6. A very simple symbolical expression for $\frac{d}{dx_{rs}}$ when $r+s>1$, *i.e.*, for any case except those of $r=0$, $s=1$ and $r=1$, $s=0$ already considered, may be obtained as follows. By (7), as in previous cases,

$$\left(\frac{dz_{20}}{dx_{rs}}\xi^2+\frac{dz_{11}}{dx_{rs}}\xi\eta+\frac{dz_{02}}{dx_{rs}}\eta^2\right)+\left(\frac{dz_{30}}{dx_{rs}}\xi^3+\ldots.\right)+\ldots$$

$$=-\frac{d\xi}{dx_{rs}}\cdot\frac{d}{d\xi}\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}$$

$$-\frac{1}{x_{01}}\eta^r\zeta^s,$$

which right-hand member may, by the second and first of equations (6), be written $$-\eta^r\zeta^s\left\{\frac{d}{d\xi}(\zeta-z_{10}\xi-z_{01}\eta)+\frac{1}{x_{01}}\right\},$$

in which inside the bracket ζ stands, for brevity, as meaning its expansion in terms of ξ and η. Now this is merely, since $z_{10} = \frac{1}{x_{01}}$,

$$-\eta^r \zeta^s \frac{d\zeta}{d\xi},$$

i.e.,

$$-\frac{1}{s+1}\eta^r \frac{d}{d\xi}(\zeta^{s+1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

which is to be taken merely as a short representation of

$$-\frac{1}{s+1}\eta^r \frac{d}{d\xi}\left[\left\{(z_{10}\xi + z_{01}\eta) + (z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2) + (z_{30}\xi^3 + \ldots) + \ldots\right\}^{s+1}\right]$$
$$\ldots\ldots\ldots\ldots (14a),$$

the differentiation being strictly partial.

The coefficient of each product $\xi^m \eta^n$, in the final form of this after expansion and differentiation, is then equal to the corresponding

$$\frac{dz_{mn}}{dx_{rs}}.$$

It is to be noticed that, as should obviously be the case, for values of n less than r, and also for values of $m+n$ less than $r+s$, this coefficient vanishes.

Hence, since

$$\frac{d}{dx_{rs}} = \sum_{m+n=1}^{m+n=\infty} \left[\frac{dz_{mn}}{dx_{rs}} \cdot \frac{d}{dz_{mn}}\right],$$

the expression for $\frac{d}{dx_{rs}}$ in terms of the operators $\frac{d}{dz_{mn}}$ is obtained by putting, for all pairs of values of the numbers m, n, $\frac{d}{dz_{mn}}$ instead of $\xi^m \eta^n$ in the final expansion in terms of ξ and η of (14); a fact which is expressed symbolically as before, by saying that

$$\frac{d}{dx_{rs}} = -\frac{1}{s+1}\eta^r \frac{d}{d\xi}(\xi^{s+1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

7. Having drawn conclusions with regard to and by means of the expressions for functions of the doubly suffixed z's, in terms of the doubly suffixed x's, let us now consider their expressions in terms of the doubly suffixed y's. The method is the same as before.

That the third of the equalities (6) is $-y_{10}$ times the first, tells us that

$$\sum_{r+s=2}^{r+s=\infty} \{z_{rs}\xi^r \eta^s\} = -\frac{1}{y_{10}} \sum_{r+s=2}^{r+s=\infty} \{y_{rs}\zeta^r \xi^s\} \ldots\ldots\ldots\ldots\ldots (16).$$

Now, in this let y_{10} alone, of all the doubly suffixed y's, receive an

infinitesimal change; and of ξ, η, ζ let, for our present purpose, ξ and ζ be kept constant, the consequent infinitesimal change of one of the three being therefore given to η. We obtain, exactly as in the beginning of §4, that, since

$$\frac{d\eta}{dy_{10}} = \zeta = \sum_{r+s=1}^{r+s=\infty} \{z_{rs}\zeta^r\eta^s\},$$

$$\sum_{r+s=2}^{r+s=\infty} \left\{\frac{dz_{rs}}{dy_{10}}\xi^r\eta^s\right\} = -z_{10}\xi\frac{d}{d\eta}\left\{\sum_{r+s=2}^{r+s=\infty}(z_{rs}\xi^r\eta^s)\right\}$$

$$-z_{21}\left[\eta\frac{d}{d\eta}+1\right]\left\{\sum_{r+s=2}^{r+s=\infty}(z_{rs}\xi^r\eta^s)\right\}$$

$$-\frac{1}{2}\frac{d}{d\eta}\left[\left\{\sum_{r+s=2}^{r+s=\infty}(z_{rs}\xi^r\eta^s)\right\}^2\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(17),$$

a result corresponding to (8). The equation of coefficients of the various terms $\xi^r\eta^s$ on the two sides of this identity gives expressions for the derivatives $\frac{dz_{rs}}{dy_{10}}$ for all combinations of zero or positive integral values of r and s whose sum exceeds unity; and to these may be added

$$\frac{dz_{10}}{dy_{10}} = \frac{y_{01}}{y_{10}^2} = -z_{10}z_{01}, \quad \text{and} \quad \frac{dz_{01}}{dy_{10}} = -\frac{1}{y_{10}^2} = -z_{01}^2,$$

obtained from $$z_{10} = -\frac{y_{01}}{y_{10}}, \quad z_{01} = \frac{1}{y_{10}}.$$

Hence, just as (10) was obtained, we deduce that, if $\xi^r\eta^s$ is adopted as a symbolical way of writing $\frac{d}{dz_{rs}}$,

$$\frac{d}{dy_{10}} = -z_{10}\left\{(z_{11}\xi^2+2z_{02}\xi\eta)+(z_{21}\xi^3+2z_{12}\xi^2\eta+3z_{03}\xi\eta^2)+\ldots\right\}$$

$$-z_{01}\left\{z_{10}\xi+z_{01}\eta+(z_{20}\xi^2+2z_{11}\xi\eta+3z_{02}\eta^2)\right.$$

$$\left.+(z_{30}\xi^3+2z_{21}\xi^2\eta+3z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}$$

$$-\frac{1}{2}\frac{d}{d\eta}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]$$

$$\ldots\ldots\ldots\ldots(18),$$

in the last expression the expansion and partial differentiation having to be effected before the substitution $\frac{d}{dz_{rs}}$, &c., for $\xi^r\eta^s$, &c., is made. In the right-hand side of (18) the operation is on any function whatever of all or any of the doubly suffixed z's, and on the left it is supposed

to be on the probably unknown equivalent function of the doubly suffixed y's. The operator on the right must then annihilate any function of the z_3 where transform does not involve y_{10}, and conversely.

Now choose, for the function of the suffixed z's operated on, an expression of the form $z_{01}^{-\mu}P$, where P is a pure function, as in (11); and express, by means of (18), the conditions that its transform is independent of y_{10}, as will in particular be the case for a proper value of μ (see later) if P is a pure reciprocant. The conclusion is that

$$-\frac{z_{10}}{z_{01}^{\mu}}\left\{\left[z_{11}\frac{d}{dz_{20}}+2z_{02}\frac{d}{dz_{11}}\right]+\left[z_{21}\frac{d}{dz_{30}}+2z_{12}\frac{d}{dz_{21}}+3z_{03}\frac{d}{dz_{12}}\right]+\ldots\right\}P$$

$$+\frac{1}{z_{01}^{\mu-1}}\left\{\mu-\left[z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+3z_{02}\frac{d}{dz_{02}}\right]\right.$$

$$\left.-\left[z_{30}\frac{d}{dz_{30}}+2z_{21}\frac{d}{dz_{21}}+3z_{12}\frac{d}{dz_{12}}+4z_{03}\frac{d}{dz_{03}}\right]-\ldots\right\}P$$

$$-\frac{1}{z_{01}^{\mu}}\cdot\frac{1}{2}\frac{d}{d\eta}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)\right.\right.$$

$$\left.\left.+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]P=0\ldots\ldots(19),$$

the last line being symbolical, as before explained.

Moreover, just as was the case with (11), this is really not one condition only, but three, the terms multiplied by

$$\frac{z_{10}}{z_{01}^{\mu}},\quad \frac{1}{z_{01}^{\mu-1}},\quad \text{and}\quad \frac{1}{z_{01}^{\mu}}$$

having necessarily to vanish separately. It is thus proved that, for the transform of $z_{01}^{-\mu}P$ to be independent of y_{10}, it is necessary, and also sufficient, that P satisfy three linear equations,

$$E_2P=0,\quad \Omega_2P=0,\quad V_2P=0,$$

where the annihilators denoted are

$$E_2\equiv\mu-\left(z_{20}\frac{d}{dz_{20}}+2z_{11}\frac{d}{dz_{11}}+3z_{02}\frac{d}{dz_{02}}\right)$$

$$-\left(z_{30}\frac{d}{dz_{30}}+2z_{21}\frac{d}{dz_{21}}+3z_{12}\frac{d}{dz_{12}}+4z_{03}\frac{d}{dz_{03}}\right)-\ldots\quad\ldots(\mathrm{D}),$$

$$\Omega_2\equiv\left(z_{11}\frac{d}{dz_{20}}+2z_{02}\frac{d}{dz_{11}}\right)+\left(z_{21}\frac{d}{dz_{30}}+2z_{12}\frac{d}{dz_{21}}+3z_{03}\frac{d}{dz_{12}}\right)+\ldots\quad\ldots\ldots(\mathrm{E}),$$

and

$$V_2 \equiv \frac{1}{2}\frac{d}{d\eta}\left[\left\{(z_{20}\xi^2+z_{11}\xi\eta+z_{02}\eta^2)+(z_{30}\xi^3+z_{21}\xi^2\eta+z_{12}\xi\eta^2+z_{03}\eta^3)+\ldots\right\}^2\right]$$

$$\equiv \frac{1}{2}\frac{d}{d\eta}\left\{(\zeta-z_{10}\xi-z_{01}\eta)^2\right\} \quad \ldots\ldots\ldots\

where the symbolism on the right means that ζ represents its expansion by Taylor's theorem in terms of ξ and η, that the expansion and partial differentiation for η are to be performed, and that in the result, for all values of m and n, $\xi^m\eta^n$ is to be replaced by $\frac{d}{dz_{mn}}$.

9. We proceed now to the theory of pure ternary reciprocants. If $R(z)$ denote such a reciprocant function of the derivatives of z with respect to x and y, and $R(x)$, $R(y)$ respectively denote the cyclically derived functions of the partial derivatives of x and of y, the expression of the property of reciprocance, viz.,

$$\frac{R(z)}{(z_{10}z_{01})^{\frac{1}{2}\mu}} = \omega^k \frac{R(x)}{(x_{10}x_{01})^{\frac{1}{2}\mu}} = \omega^{2k} \frac{R(y)}{(y_{10}y_{01})^{\frac{1}{2}\mu}} \quad \text{.....(23),}$$

may be written (Vol. XVII., p. 184, Note), more suitably for our present purpose,

$$\frac{R(z)}{(-z_{10})^\mu} = \omega^k R(x), \quad \frac{R(z)}{(-z_{01})^\mu} = \omega^{2k} R(y) \quad \text{.....(24).}$$

A pure ternary reciprocant is then a function which unites in itself the properties of the pure functions P considered in §§ 4 and 7. It will be seen later that, conversely, all pure functions P which possess these united properties are reciprocants, or at least sums of reciprocants, of like index μ, in case there be more than one such. The facts now obtained with regard to pure ternary reciprocants may be stated as follows:—

(i.) *A pure ternary reciprocant is necessarily a full invariant of the system of emanants*

$$(z_{20}, z_{11}, z_{02})(\xi, \eta)^2, \text{ \&c.}$$

This was proved in my former paper (Vol. XVII., p. 181); and it is to be noticed that the proof there given requires that it be a single invariant, being thus more stringent than that here given of its annihilation by the Ω_1 and Ω_2 of results (B) and (E), which would permit of its being a sum of different invariants of the emanants.

One result of the fact that the reciprocant is a single invariant is that it must be necessarily homogeneous. Thus,

(ii.) *It is homogeneous and doubly isobaric, having each of its terms of the same constant weight in both first and second suffixes.*

The double isobarism, as well as the homogeneity, really follows from the fact of the reciprocant being a single invariant. Granted, however,

the homogeneity, the constancy throughout of the two partial weights follows from the fact of annihilation by the E_1 and E_2 of (A) and (D), while the equality of each to the half of the full weight or sum of suffixes arises from the fact that the μ is the same both in E_1 and E_2, by (24). Hence, also,

(iii.) *If i be the degree and w the full weight of a pure ternary reciprocant, the index μ of that reciprocant is given by*

$$\mu = i + \tfrac{1}{2}w.$$

It is, of course, to be remarked that the full weight (sum of suffixes) w is always even.

To these facts we shall be enabled, by the next article, to add—

(iv.) *The mark of character k of a pure ternary reciprocant is always zero.*

But the property of pure ternary reciprocants, to which all these other facts are merely subsidiary, is that,

(v.) *They have two quadro-linear annihilators V_1 and V_2.*

For these symbolical expressions have been found in results (C) and (F). The following will be found to be their expanded forms, written in ordinary notation :—

$$\begin{aligned}
V_1 \equiv\ & 2z_{20}^2 \frac{d}{dz_{30}} + 3z_{20}z_{11} \frac{d}{dz_{21}} + 2\,(z_{20}z_{02} + \tfrac{1}{2}z_{11}^2) \frac{d}{dz_{12}} + z_{11}z_{02} \frac{d}{dz_{03}} \\
& + 5z_{20}z_{30} \frac{d}{dz_{40}} + 4\,(z_{20}z_{21} + z_{11}z_{30}) \frac{d}{dz_{31}} + 3\,(z_{20}z_{12} + z_{11}z_{21} + z_{02}z_{30}) \frac{d}{dz_{22}} \\
& + 2\,(z_{20}z_{03} + z_{11}z_{12} + z_{02}z_{21}) \frac{d}{dz_{13}} + (z_{11}z_{03} + z_{02}z_{12}) \frac{d}{dz_{04}} \\
& + 6\,(z_{20}z_{40} + \tfrac{1}{2}z_{30}^2) \frac{d}{dz_{50}} + 5\,(z_{20}z_{31} + z_{11}z_{40} + z_{30}z_{21}) \frac{d}{dz_{41}} \\
& + 4\,(z_{20}z_{22} + z_{11}z_{31} + z_{02}z_{40} + \tfrac{1}{2}z_{21}^2 + z_{12}z_{30}) \frac{d}{dz_{32}} \\
& + 3\,(z_{20}z_{13} + z_{11}z_{22} + z_{02}z_{31} + z_{21}z_{12} + z_{30}z_{03}) \frac{d}{dz_{23}} \\
& + 2\,(z_{20}z_{04} + z_{11}z_{13} + z_{02}z_{22} + \tfrac{1}{2}z_{12}^2 + z_{21}z_{03}) \frac{d}{dz_{14}} \\
& + (z_{11}z_{04} + z_{02}z_{13} + z_{12}z_{03}) \frac{d}{dz_{05}}
\end{aligned}$$

$$+7\,(z_{20}z_{50}+z_{30}z_{40})\,\frac{d}{dz_{60}}+6\,(z_{20}z_{41}+z_{11}z_{50}+z_{30}z_{31}+z_{21}z_{40})\,\frac{d}{dz_{51}}$$

$$+5\,(z_{20}z_{32}+z_{11}z_{41}+z_{02}z_{50}+z_{30}z_{22}+z_{21}z_{31}+z_{12}z_{40})\,\frac{d}{dz_{42}}$$

$$+4\,(z_{20}z_{23}+z_{11}z_{32}+z_{02}z_{41}+z_{30}z_{13}+z_{21}z_{22}+z_{12}z_{31}+z_{03}z_{40})\,\frac{d}{dz_{33}}$$

$$+3\,(z_{20}z_{14}+z_{11}z_{23}+z_{02}z_{32}+z_{30}z_{04}+z_{21}z_{13}+z_{12}z_{22}+z_{03}z_{31})\,\frac{d}{dz_{24}}$$

$$+2\,(z_{20}z_{05}+z_{11}z_{14}+z_{02}z_{23}+z_{21}z_{04}+z_{12}z_{13}+z_{03}z_{22})\,\frac{d}{dz_{15}}$$

$$+(z_{11}z_{05}+z_{02}z_{14}+z_{12}z_{04}+z_{03}z_{13})\,\frac{d}{dz_{06}}$$

$$+\ldots\ldots$$

$$+\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C');$$

$$V_2 \equiv z_{11}z_{20}\,\frac{d}{dz_{30}}+2\,(\tfrac{1}{2}z_{11}^2+z_{03}z_{20})\,\frac{d}{dz_{21}}+3z_{11}z_{02}\,\frac{d}{dz_{12}}+2z_{02}^2\,\frac{d}{dz_{03}}$$

$$+(z_{11}z_{30}+z_{20}z_{21})\,\frac{d}{dz_{40}}+2\,(z_{02}z_{30}+z_{11}z_{21}+z_{20}z_{12})\,\frac{d}{dz_{31}}$$

$$+3\,(z_{02}z_{21}+z_{11}z_{12}+z_{20}z_{03})\,\frac{d}{dz_{22}}+4\,(z_{02}z_{12}+z_{11}z_{03})\,\frac{d}{dz_{13}}+5z_{02}z_{03}\,\frac{d}{dz_{04}}$$

$$+(z_{11}z_{40}+z_{20}z_{31}+z_{21}z_{30})\,\frac{d}{dz_{50}}$$

$$+2\,(z_{02}z_{40}+z_{11}z_{31}+z_{20}z_{22}+\tfrac{1}{2}z_{21}^2+z_{12}z_{30})\,\frac{d}{dz_{41}}$$

$$+3\,(z_{02}z_{31}+z_{11}z_{22}+z_{20}z_{13}+z_{12}z_{21}+z_{03}z_{30})\,\frac{d}{dz_{32}}$$

$$+4\,(z_{02}z_{22}+z_{11}z_{13}+z_{20}z_{04}+\tfrac{1}{2}z_{12}^2+z_{21}z_{03})\,\frac{d}{dz_{23}}$$

$$+5\,(z_{02}z_{13}+z_{11}z_{04}+z_{03}z_{12})\,\frac{d}{dz_{14}}$$

$$+6\,(z_{02}z_{04}+\tfrac{1}{2}z_{03}^2)\,\frac{d}{dz_{05}}$$

$$+(z_{11}z_{50}+z_{20}z_{41}+z_{30}z_{31}+z_{21}z_{40})\,\frac{d}{dz_{60}}$$

$$+2\,(z_{02}z_{50}+z_{11}z_{41}+z_{20}z_{32}+z_{12}z_{40}+z_{21}z_{31}+z_{30}z_{22})\,\frac{d}{dz_{51}}$$

$$+3\,(z_{02}z_{41}+z_{11}z_{32}+z_{20}z_{23}+z_{03}z_{40}+z_{12}z_{31}+z_{21}z_{22}+z_{30}z_{13})\,\frac{d}{dz_{42}}$$

$$+4\,(z_{02}z_{32}+z_{11}z_{23}+z_{20}z_{14}+z_{03}z_{31}+z_{12}z_{22}+z_{21}z_{13}+z_{30}z_{04})\,\frac{d}{dz_{33}}$$

$$+5\,(z_{02}z_{23}+z_{11}z_{14}+z_{20}z_{05}+z_{03}z_{22}+z_{12}z_{13}+z_{21}z_{04})\,\frac{d}{dz_{24}}$$

$$+6\,(z_{02}z_{14}+z_{11}z_{05}+z_{03}z_{13}+z_{12}z_{04})\,\frac{d}{dz_{15}}$$

$$+7\,(z_{02}z_{05}+z_{03}z_{04})\,\frac{d}{dz_{06}}$$

$$+\ldots\ldots$$

$$+\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{F}').$$

10. Let us now proceed to the proof that the conditions (i.), (ii.), (iii.), (v.) of § 9, which have been found as such that a pure ternary reciprocant must satisfy, are conversely such that functions of the second and higher derivatives which satisfy them are necessarily reciprocants.

The formulæ for transforming a partial differential expression in x, y, z, from a form in which x and y are independent variables to one in which y and z are independent, are

$$z_{10}=\frac{1}{x_{01}},\quad z_{01}=-\frac{x_{10}}{x_{01}},$$

$$\frac{d}{dx}=\frac{1}{x_{01}}\,\frac{d}{dz},\quad \frac{d}{dy}=\frac{d}{d}-\frac{x_{10}}{x_{01}}\,\frac{d}{dz}.$$

Thus, if r and s be numbers (inclusive of zero) whose sum exceeds unity,

$$\frac{d^{r+s}z}{dx^r dy^s}=\frac{d^{r+s-1}}{dx^{r-1}dy^s}(z_{10})$$

$$=\left(\frac{1}{x_{01}}\,\frac{d}{dz}\right)^{r-1}\left(\frac{d}{dy}-\frac{x_{10}}{x_{01}}\,\frac{d}{dz}\right)^{s}\left(\frac{1}{x_{01}}\right)$$

$$=\frac{1}{x_{01}^{r-1}}\,\frac{d^{r+s-1}}{dz^{r-1}dy^s}\left(\frac{1}{x_{01}}\right)+\ldots$$

$$=-\frac{1}{x_{01}^{r+1}}\,\frac{d^{r+s}x}{dy^s dz^r}+\ldots,$$

the terms not written down all involving powers of $\frac{1}{x_{01}}$ surpassing

the $(r+1)^{\text{th}}$. We have, then, after multiplication by $\frac{1}{r\,!\,s\,!}\,x_{01}^{r+1}$,

$$\frac{z_{rs}}{z_{10}^{r+1}} = -x_{sr} + \text{terms with } \frac{1}{x_{01}} \text{ as a factor} \quad \ldots\ldots\ldots\ldots(25),$$

in which it is to be noticed that the index of the power of z_{10} in the denominator on the left exceeds r, the first suffix-weight of its numerator, by 1, the degree of that numerator.

Now, take $H(z)$ a homogeneous function of degree i of the derivatives z_{rs}, such as to satisfy condition (ii.) of § 9, *i.e.*, to be isobaric and of the same partial weight $\frac{1}{2}w$ in first and second suffixes separately, and consequently to be of full weight w. Let $H(x)$ be the result of replacing every z_{rs} in it by the corresponding x_{rs}, and let $H'(x)$ be the result of replacing in $H(x)$ every derivative x_{rs} by its conjugate with suffixes interchanged x_{sr}. From (25), we obtain

$$\frac{H(z)}{z_{10}^{i+\frac{1}{2}w}} = (-1)^i H'(x) + \text{terms involving } x_{01}.$$

Let, further, $H(z)$ have been so chosen as to be annihilated by Ω_1 and V_1. By § 4, the expression for the left-hand side of the foregoing equality in terms of suffixes x's must, this being so, be independent of x_{01}; and we must have, consequently,

$$\frac{H(z)}{z_{10}^{i+\frac{1}{2}w}} = (-1)^i H'(x).$$

But, again, let $H(z)$ be subject to annihilation by Ω_2 as well as by Ω_1. It will then, being homogeneous and doubly isobaric, be a single full invariant of the system of emanants, and must consequently be unaltered (or altered only in sign, if a skew-invariant) when each coefficient z_{rs} in every emanant is replaced by its conjugate coefficient z_{sr}, *i.e.*, when first and second suffixes are interchanged throughout. Thus, writing x instead of z,

$$H'(x) = (-1)^{\frac{1}{2}w} H(x),$$

an invariant being skew or not according as its weight ($\frac{1}{2}w$ in this case) is or is not odd. Hence, finally,

$$\frac{H(z)}{z_{10}^{i+\frac{1}{2}w}} = (-1)^{i+\frac{1}{2}w} H(x) \quad \ldots\ldots\ldots\ldots(26);$$

so that, in virtue of (24), $H(z)$ has the property of a reciprocant, so

far as the first transformation from z dependent variable to x dependent is concerned.

In exactly the same way it follows that, if $H(z)$ is annihilated by V_2 as well as by Ω_1 and Ω_2,

$$\frac{H(z)}{z_{01}^{i+\frac{1}{2}w}} = (-1)^{i+\frac{1}{2}w} H(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27);$$

which, by (24), is the only further necessity for $H(z)$ to be a pure ternary reciprocant. The comparison of (26) and (27) with (24) makes evident the accuracy of the statement (iv.) of § 9, that the mark of character k of a pure ternary reciprocant is of necessity zero.

The sufficiency, as well as the necessity, of the conditions expressed in § 9, for a pure function R to be a ternary reciprocant, has then been fully established.

11. That

$$z_{20}z_{02} - \tfrac{1}{4}z_{11}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

the discriminant of the second emanant, is a reciprocant, I have already proved from first principles [Vol. XVII., p. 180, (30)]. That it satisfies the determining conditions arrived at in this paper, is so immediately obvious as only to require statement.

A second pure ternary reciprocant is

$$\begin{vmatrix} z_{30}, & z_{21}, & z_{12}, & z_{03} & \\ & z_{30}, & z_{21}, & z_{12}, & z_{03} \\ z_{20}, & z_{11}, & z_{02} & & \\ & z_{20}, & z_{11}, & z_{02} & \\ & & z_{20}, & z_{11}, & z_{02} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

the discriminant of the quadratic and cubic emanants. That it must be one is necessary, if the fact be granted that, to use a convenient nomenclature of Professor Sylvester's, it is the criterion of ruled surfaces, which can, of course, have no special respect to any axes of reference. (A remark akin to this has been made by M. Halphen, at the end of his *Thèse sur les Invariants Différentiels.*) The same may also, without difficulty, be shown from elementary considerations; and it may be here remarked that like reasoning will apply to the eliminant of three emanants in the matter of quaternary reciprocants, that of four of quinary, and generally that of the second to the n^{th} emanants in the matter of n-ary reciprocants.

That the eliminant (29) is annihilated by V_1 and V_2, and so obeys the laws arrived at in this paper, is readily seen. The operation of

V_1 upon it gives

$$\begin{vmatrix} 2z_{20}^2, & 3z_{20}z_{11}, & 2z_{20}z_{02}+z_{11}^2, & z_{11}z_{02} & \\ z_{30}, & z_{21}, & z_{12}, & z_{03} \\ z_{20}, & z_{11}, & z_{02} & & \\ z_{20}, & z_{11}, & z_{02} & & \\ & z_{20}, & z_{11}, & z_{02} & \end{vmatrix}$$

$$+ \begin{vmatrix} z_{30}, & z_{21}, & z_{12}, & z_{03} & \\ 2z_{20}^2, & 3z_{20}z_{11}, & 2z_{20}z_{02}+z_{11}^2, & z_{11}z_{02} & \\ z_{20}, & z_{11}, & z_{02}, & & \\ z_{20}, & z_{11}, & z_{02} & & \\ & z_{20}, & z_{11}, & z_{02} & \end{vmatrix}$$

Now, in the first of these determinants, the first row is the sum of $2z_{20}$ times the third and z_{11} times the fourth; and in the second, the second row is the sum of $2z_{20}$ times the fourth and z_{11} times the fifth; so that both vanish. In like manner, V_2 is an annihilator of (29). That the other laws of § 9 are obeyed, is clear in virtue of the properties of eliminants as invariants.

12. A third pure ternary reciprocant will now be determined. We know (see, for instance, Salmon's *Higher Plane Curves*, 2nd ed., § 193) that the quadratic and quartic emanants have an invariant of degree 3, and partial weights 4, 4,

$$I(a^2c) = 6(z_{02}^2z_{40}+z_{20}^2z_{04})-3z_{11}(z_{02}z_{31}+z_{20}z_{13})+z_{22}(z_{11}^2+2z_{02}z_{20}),$$

and that the cubic and quadratic emanants have one of like type,

$$I(ab^2) = 2z_{02}(3z_{30}z_{12}-z_{21}^2)-z_{11}(9z_{30}z_{03}-z_{12}z_{21})+2z_{20}(3z_{03}z_{21}-z_{12}^2).$$

For any constant value of k, then,

$$I(a^2c)+kI(ab^2)$$

is an invariant of the quadratic cubic and quartic emanants satisfying the laws (i.), (ii.), (iii.) of § 9. Now it is at once found that

$$V.I(a^2c) = 36z_{02}^2z_{20}z_{30}-9z_{02}z_{11}^2z_{30}-12z_{02}z_{11}z_{20}z_{21}+12z_{02}z_{20}^2z_{12}$$
$$+3z_{11}^3z_{21}-3z_{11}^2z_{20}z_{12},$$

and that

$$V.I(ab^2) = 12z_{02}^2 z_{20} z_{30} - 3z_{02} z_{11}^2 z_{20} - 4z_{02} z_{11} z_{20} z_{21} + 4z_{02} z_{20}^2 z_{12} + z_{11}^3 z_{21} - z_{11}^2 z_{20} z_{12},$$

the term $z_{11} z_{20}^2 z_{03}$ not occurring in either.

Now, the first of these right-hand members is three times the second. It follows that

$$I(a^2c) - 3I(ab^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

is annihilated by V_1. In like manner, V_2 annihilates it. Consequently it is a reciprocant.

In (28), (29), and (30), we have then three pure ternary reciprocants. From these we know two that are independent and absolute, obtained by dividing either two of the three by proper powers of the third. Hence, by my former paper [Vol. XVII., p. 183, (34)], we have the means of educing an infinite series of pure ternary reciprocants by simple processes of differentiation.

Note on Two Annihilators in the Theory of Elliptic Functions.

By JOHN GRIFFITHS, M.A.

[*Read Dec. 9th*, 1886.]

NOTATION.

$P(x, k)$ and $Q(x, k)$ are taken to be algebraic functions of x involving k-functions—not necessarily rational and integral functions of x. [For example, we may have

$$P = (1+k')\, x\sqrt{1-x^2}, \quad Q = \sqrt{1-k^2x^2}.]$$

$$R^2 = Q^2 - \lambda^2 P^2, \quad S^2 = Q^2 - P^2.$$

$$\partial_k = \frac{d}{dk}, \quad \partial_x = \frac{d}{dx}.$$

$$\Omega = nkk'^2\partial_k + (1-x^2)(1-k^2x^2)\,\phi(x, k)\,\partial_x,$$

$$O = nkk'^2\partial_k + (1-x^2)(1-k^2x^2)\,f(x, k)\,\partial_x,$$

where $\phi(x, k) = \partial_x \log\{R \div (1-k^2x^2)^{\frac{1}{n}}\} = \dfrac{nk^2x}{1-k^2x^2} + \dfrac{1}{R}\partial_x R,$

$$f(x, k) = \partial_x \log\{S \div (1-k^2x^2)^{\frac{1}{n}}\} = \frac{nk^2x}{1-k^2x^2} + \frac{1}{S}\partial_x S,$$

n a number, and $k^2+k'^2 = 1$, $\lambda^2+\lambda'^2 = 1$.

I. *On the Annihilators* Ω *and* O.

Every algebraic elliptic transformation equation $y = \dfrac{P}{Q}$ which gives

$$\frac{dy}{\sqrt{1-y^2 \cdot 1-\lambda^2y^2}} = M\frac{dx}{\sqrt{1-x^2 \cdot 1-k^2x^2}},$$

$$\Theta\{M(u+a)+\Lambda, \lambda\} = Ce^{\mu u^2+\nu u}\Theta^n u \cdot R(\operatorname{sn} u),$$

$$\Theta\{M(u+a), \lambda\} = C\sqrt{\lambda'}\, e^{\mu u^2+\nu u}\Theta^n u \cdot Q(\operatorname{sn} u),$$

and $$M^2 = n\frac{kk'^2}{\lambda\lambda'^2}\frac{d\lambda}{dk},$$

has two annihilators Ω and O; *i.e.*, if Ω is applied to y or $\dfrac{P}{Q}$, then

$$\Omega y = \Omega\frac{P}{Q} = 0;$$

and, consequently, as I shall prove,

$$O\lambda y = O\lambda\frac{P}{Q} = 0. \quad \text{(See Notation.)}$$

To illustrate the formulæ, I take Landen's equation

$$y = \frac{P}{Q} = \frac{(1+k')\,x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}.$$

Here $$P = (1+k')\,x\sqrt{1-x^2}, \quad Q = \sqrt{1-k^2x^2},$$

$$\lambda = \frac{1-k'}{1+k'}, \quad n = 2,$$

$$R = \sqrt{Q^2-\lambda^2P^2} = 1-(1-k')\,x^2;$$

so that $\phi(x, k) = \dfrac{nk^2x}{1-k^2x^2} + \dfrac{1}{R}\partial_x R = \dfrac{2k^2x}{1-k^2x^2} - \dfrac{2(1-k')\,x}{1-(1-k')\,x^2},$

$$\Omega = 2kk'^2\partial_k + (1-x^2)(1-k^2x^2)\,\phi\,(x, k)\,\partial_x$$

$$= 2kk'^2\partial_k + (1-x^2)\,\frac{2k'\,(1-k')\,x}{1-(1-k')\,x^2}\,\partial_x,$$

$$\partial_k y = -\,\frac{k}{k'}\,\frac{x\,.\,1-(1+k')\,x^2\,.\,(1-x^2)^{\frac{1}{2}}}{(1-k^2x^2)^{\frac{3}{2}}},$$

$$\partial_x y = 1+k'\,.\,1-(1+k')\,x^2\,.\,1-(1-k')\,x^2 \div (1-x^2)^{\frac{1}{2}}\,(1-k^2x^2)^{\frac{3}{2}},$$

and, consequently, $\Omega y = 0.$

II. *Other Forms of* Ω *and* O.

The annihilator Ω was brought to my notice in the course of some work connected with the transformation of Θ-functions. The proof that

$$\Omega\,\frac{P}{Q} = 0$$

is not difficult, but I do not propose to discuss it in this Note.

If we write

$$U = nkk'^2P\partial_k + (1-x^2)\,\{nk^2xP + (1-k^2x^2)\,\partial_x P\}\,\partial_x,$$

$$V = nkk'^2Q\partial_k + (1-x^2)\,\{nk^2xQ + (1-k^2x^2)\,\partial_x Q\}\,\partial_x,$$

then $\Omega_1 = QV - \lambda^2 PU, \qquad O_1 = QV - PU,$

$$= (Q^2 - \lambda^2 P^2)\,\Omega \qquad = (Q^2 - P^2)\,O.$$

A third form is obtained by putting $x = \operatorname{sn} u$; viz.,

$$\Omega_2 = nkk'^2\partial_k + \left\{ nk^2\,\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} + \frac{1}{R}\,\partial_u R \right\}\partial_u,$$

where $R^2 = Q^2 - \lambda^2 P^2, \quad P \equiv P\,(\operatorname{sn} u, k), \quad Q \equiv Q\,(\operatorname{sn} u, k).$

III. *Some Properties of the Operators* U *and* V.

(1) If $y = \dfrac{P}{Q}$, and U be the operator in II., then

$$Uy = \frac{M^2}{Q}\,(Q^2 - P^2).$$

Here $Uy = nkk'^2P\partial_k y + (1-x^2)\,\{nk^2xP + (1-k^2x^2)\,\partial_x P\}\,\partial_x y,$

and, since $\Omega y = 0,$

i.e., $$nkk'^2\partial_k y+(1-x^2)(1-k^2x^2)\,\phi\,(x,k)\,\partial_x y=0,$$

where $$\phi\,(x,k)=\frac{nk^2x}{1-k^2x^2}+\frac{1}{R}\,\partial_x R,$$

we have $$Uy=(1-x^2)(1-k^2x^2)\left\{\partial_x P-\frac{P}{R}\,\partial_x R\right\}\partial_x y$$

$$=(1-x^2)(1-k^2x^2)\,R\left\{\partial_x\frac{P}{R}\right\}\partial_x y.$$

But $$\frac{P}{R}=\frac{y}{\sqrt{1-\lambda^2y^2}},\quad \partial_x\frac{P}{R}=(1-\lambda^2y^2)^{-\frac{3}{2}}\partial_x y,$$

$$\partial_x y=M\sqrt{\frac{1-y^2\,.\,1-\lambda^2y^2}{1-x^2\,.\,1-k^2x^2}}.$$

Hence, finally, $$Uy=U\frac{P}{Q}=\frac{M^2}{Q}\,(Q^2-P^2),$$

or $$QUP-PUQ=M^2Q\,(Q^2-P^2);$$

so that UQ produces the factor Q.

The following are some other results as regards U and V:—

(2) $$QVP-PVQ=M^2\lambda^2P\,(Q^2-P^2);$$

i.e., VP produces the factor P.

(3) $$QU\lambda P-\lambda PUQ=M^2\lambda Q\,(Q^2-\lambda^2P^2).$$

(4) $$QV\lambda P-\lambda PVQ=M^2\lambda P\,(Q^2-\lambda^2P^2).$$

(5) $$\frac{V\lambda y}{U\lambda y}=y=\frac{P}{Q}=\frac{\Omega P}{\Omega Q}=\frac{O\lambda P}{\lambda OQ}=\frac{Vy}{\lambda^2Uy}.$$

(6) $$O_1y=-M^2\lambda'^2PQ\,(1-y^2),\quad Oy=-M^2\lambda'^2y.$$

(7) $$\Omega_1\lambda y=M^2\lambda\lambda'^2PQ\,(1-\lambda^2y^2),\quad \Omega\lambda y=M^2\lambda\lambda'^2y.$$

(8) $$\frac{\lambda O_1y}{\Omega_1\lambda y}=-\frac{O_1\lambda}{\Omega_1\lambda},$$

where $$M^2=n\frac{kk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk},$$

$$\Omega_1=(Q^2-\lambda^2P^2)\,\Omega,\quad O_1=(Q^2-P^2)\,O.$$

The above examples are sufficient to illustrate the operators U, V, and the annihilators Ω and O.

IV.

If $\Omega y = 0$, then $O\lambda y = 0$.*

Otherwise.—If $\Omega_1 y = 0$, then $O_1\lambda y = 0$.

Here $$\Omega_1 = QV - \lambda'^2 PU, \quad O_1 = QV - PU,$$

and, consequently, $$\Omega_1 = O_1 + \lambda'^2 PU.$$

Hence, since $\Omega_1 y$ is by hypothesis zero, we have

$$O_1 y + \lambda'^2 PUy = 0.$$

Now, $$O_1\lambda y = \lambda O_1 y + y O_1\lambda.$$

Therefore, if $O_1\lambda y = 0$, then $\lambda\lambda'^2 PUy = y O_1\lambda$,

or
$$\begin{aligned}\lambda\lambda'^2 PUy &= y\,(QV\lambda - PU\lambda)\\ &= y\,(Q^2 - P^2)\,nkk'^2\frac{d\lambda}{dk}\\ &= M^2\lambda\lambda'^2 y\,(Q^2 - P^2),\end{aligned}$$

$$Uy = \frac{M^2}{Q}(Q^2 - P^2),$$

the result already proved in III.

V. *An example of the application of the Annihilator* Ω *to the Theory of Composition or Addition.*

I take the rational transformation-equation

$$y = \sin(\theta + A + B + C + \ldots) = \frac{P}{Q},$$

where $$\sin\theta = x,$$

$$\sin A = \frac{2\alpha' x\sqrt{1-x^2}}{1-\alpha^2 x^2}, \quad \cos A = \frac{1-(1+\alpha')\,x^2}{1-\alpha^2 x^2},$$

$$\sin B = \frac{2\beta' x\sqrt{1-x^2}}{1-\beta^2 x^2}, \quad \cos B = \frac{1-(1+\beta')\,x^2}{1-\beta^2 x^2}, \text{ \&c.,}$$

$$\alpha^2 + \alpha'^2 = 1, \quad \beta^2 + \beta'^2 = 1,$$

and the coefficients α, β, ... are k-functions.

Here $$\partial_k\theta = 0, \quad (1-x^2)\,\partial_x\theta = \sqrt{1-x^2},$$

* Conversely, the two conditions $\Omega y = 0$, $O\lambda y = 0$ are not only necessary but sufficient for the Θ transformation mentioned in Sect. I.

$$\partial_k A = \frac{2x\sqrt{1-x^2}}{1-a^2x^2}\,\frac{da'}{dk}, \quad (1-x^2)\,\partial_x A = \frac{2a'\sqrt{1-x^2}}{1-a^2x^2},$$

$$\partial_k B = \quad \ldots \quad \ldots, \quad \ldots \quad \ldots \quad \ldots \quad \ldots;$$

$$\Omega A = nkk'^2\partial_k A + (1-x^2)(1-k^2x^2)\,\phi\,(x,k)\,\partial_x A,$$

where $$\phi\,(x,k) = \partial_x \log\left\{R \div (1-k^2x^2)^{\frac{1}{2}n}\right\},$$

or $$\Omega A = \frac{2nkk'^2\sqrt{1-x^2}}{1-a^2x^2}\,\frac{da'}{dk} + \sqrt{1-x^2}\,(1-k^2x^2)\,\phi\,(x,k)\,\frac{2a'}{1-a^2x^2}.$$

Consequently $$\Omega\,(\theta + A + B + \ldots) = 0,$$

if $$2nkk'^2\,\Sigma\frac{x}{1-a^2x^2}\,\frac{da'}{dk} + (1-k^2x^2)\,\phi\,(x,k)\left\{1+\Sigma\frac{2a'}{1-a^2x^2}\right\} = 0,$$

$$\phi\,(x,k) = \partial_x \log\left\{R \div (1-k^2x^2)^{\frac{1}{2}n}\right\}.$$

Now, it has been shown in a paper by the present writer (see *Proc. Lond. Math. Soc.*, Vol. XVI., p. 101) that the method of composition gives the identity

$$-nk\,\Sigma\frac{1}{1-a^2x^2}\,\frac{da'}{dk} = \left\{1+\Sigma\frac{2a'}{1-a^2x^2}\right\}\left\{\Sigma\frac{a^2}{1-a^2-(k^2-a^2)\,x^2}\right\}.$$

Hence $$\Omega\,(\theta + A + B + \ldots) = 0,$$

if $$\phi\,(x,k) = \frac{2k'^2}{1-k^2x^2}\,\Sigma\frac{a^2x}{1-a^2+(a^2-k^2)\,x^2},$$

$$\partial_x \log\left\{R \div (1-k^2x^2)^{\frac{1}{2}n}\right\} = 2\Sigma\left\{\frac{k^2x}{1-k^2x^2} + \frac{(a^2-k^2)\,x}{1-a^2+(a^2-k^2)\,x^2}\right\};$$

i.e., finally, by integration, if

$$R = (1-k^2x^2)^{\frac{1}{2}}\left[1-\frac{k^2-a^2}{1-a^2}\,x^2\right].$$

The result, $R = \sqrt{1-k^2x^2} \times$ product of factors of the form in question, is, of course, well known, and it thus appears that

$$\Omega\left\{\theta + \Sigma A\right\} = 0, \text{ or } \Omega y = 0,$$

i.e., Ω is an annihilator for Jacobi's transformation equations. Hence, also, from IV., $O\lambda y = 0$.

There would be no difficulty in applying Ω to the irrational equation

$$y = \sin\,(L + A + B + \ldots),$$

see *Proc. Lond. Math. Soc.*, Vol. XVII., p. 12.

January 13th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Prof. G. B. Mathews, B.A., University of North Wales, Bangor, was elected a Member.

The Auditor (Mr. Heppel) made his report; upon the motion of Mr. S. Roberts, seconded by Mr. J. J. Walker, the Treasurer's report was accepted, and a vote of thanks unanimously accorded to the Auditor.

The following communications were made:—

Conjugate Tucker Circles: R. Tucker.

On the Incorrectness of the Rules for contracting the processes of finding the Square and Cube Roots of a Number: Prof. M. J. M. Hill.

On the Complex Angle: J. J. Walker.

Communications were also made (with a view to elicit special information) by Messrs. Heppel, MacMahon, and S. Roberts, in the discussion of which several members took part.

The following presents were received:—

Carte-de-Visite Likeness, from Mr. C. Leudesdorf.

"Proceedings of the Royal Society," Vol. XLI., No. 247.

"The Nautical Almanac," for 1890.

"The Educational Times," for January.

"Bulletin des Sciences Mathématiques," Tome X., Dec., 1886.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 9, 10, 11.

"Beiblätter zu den Annalen der Physik und Chemie," Band X., St. 11, 12.

"Jahrbuch über die Fortschritte der Mathematik," Band XVI., Heft 1; Jahrgang 1884.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VII., No. 3.

"Bollettino delle Pubblicazioni Italiane," Nos. 23 and 24; Firenze, 1886.

"Niew Archief voor Wiskunde," Deel XIII., St. 1.

"Acta Mathematica," 9, 2.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," Jahr. XXX., Heft 1—4; Jahr. XXXI., Heft 1, 2.

"Sur une Quartique Unicursale" (*Jour. de Math. Spéciales*, 1886); "Sur les Sous-invariants des Formes Binaires," 1886; "De la Déviation dans l'Ellipse" (*Nouv. Ann.*); "Sur certaines Suites de Fractions Irréductibles," Bruxelles, 1886; from M. M. D'Ocagne.

On the Incorrectness of the Rules for contracting the processes of finding the Square and Cube Roots of a Number.

By M. J. M. Hill, Professor of Mathematics at University College, London.

[*Read Jan. 13th*, 1887.]

I. *The rule for contracting the process of finding the Square Root.*

The rule (see *Todhunter's Algebra*, Art. 246) is this :—

> When $n+1$ figures of a square root have been obtained by the ordinary method, n more may be obtained by division only, supposing $2n+1$ to be the whole number.

It will be shown in this paper that the rule cannot always be extended to cases in which the $2n+1$ figures are followed by other figures, as is usually assumed.

Let the number be denoted by N; let its square root consist of $2n+1$ figures followed by a decimal. Let the first $n+1$ figures followed by n zeros be denoted by a. Let the remaining n figures be denoted by b, and the decimal part by c.

Therefore $$\sqrt{N} = a+b+c,$$

therefore $$\frac{N-a^2}{2a} = b+c+\frac{(b+c)^2}{2a}.$$

If it can be shown that $c+\frac{(b+c)^2}{2a}$ may be greater than 1, but not so great as 2, the result of the division $\frac{N-a^2}{2a}$ may have an integral part $b+1$, and not b; and in such cases the rule will fail.

Putting $u = c+\frac{(b+c)^2}{2a}$; since a, b, c are all positive, it follows that u increases when a diminishes, but when b and c increase.

Now the least value of a is 10^{2n}; and the greatest value of b is when it consists of n nines, and is therefore 10^n-1. The greatest value c can have must be less than 1 : let this be denoted by $\cdot\dot{9}$. Consequently, the greatest value of u is

$$\cdot\dot{9}+\frac{(10^n-1+\cdot\dot{9})^2}{2\,.\,10^{2n}},$$

which approaches indefinitely close to $1\frac{1}{2}$.

Consequently the rule does not always furnish a correct result.

Two Propositions (A.) and (B.) will now be proved.

(A.) If the nearest integer to $\sqrt{N-a}$ is b, then $\frac{N-a^2}{2a}$ is $< b+1$, so that the rule is correct in this case.

But the nearest integer to $\frac{N-a^2}{2a}$ may be

(1) b ..... [Case A.1],

or (2) $b+1$ ..... [Case A.2].

In this proposition

$$0 < c < \tfrac{1}{2},$$

therefore

$$\begin{aligned} 0 < c + \frac{(b+c)^2}{2a} &< \tfrac{1}{2} + \frac{(b+\frac{1}{2})^2}{2a} \\ &< \tfrac{1}{2} + \frac{(10^n - 1 + \frac{1}{2})^2}{2.10^{2n}} \\ &< \tfrac{1}{2} + \tfrac{1}{2}\left(1 - \tfrac{1}{2}\cdot\frac{1}{10^n}\right)^2 \\ &< \tfrac{1}{2} + \tfrac{1}{2} \\ &< 1. \end{aligned}$$

Consequently $c + \frac{(b+c)^2}{2a}$ can have values ranging between 0 and 1. It can reach the lower limit when $b = 0$, $c = 0$; but it cannot come very close to the upper limit unless n be large.

All that is necessary to observe is that $c + \frac{(b+c)^2}{2a}$ may be either less or greater than $\frac{1}{2}$, so that the nearest integer to $\frac{N-a^2}{2a}$ may be b or $b+1$.

Numerical Examples.

Case A.1 :

$$N = 11884,$$

$$\sqrt{N} = 109{\cdot}01\ldots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N} - a = 9{\cdot}01\ldots,$$

$$\frac{N-a^2}{2a} = 9{\cdot}42.$$

The nearest integer to $\sqrt{N-a}$ is in this case b;
the integral part of $\frac{N-a^2}{2a}$ is also b;
and the nearest integer to $\frac{N-a^2}{2a}$ is also b.

Case A.2: $$N = 11903,$$
$$\sqrt{N} = 109{\cdot}1\ldots,$$
$$a = 100,$$
$$b = 9,$$
$$\sqrt{N-a} = 9{\cdot}1\ldots,$$
$$\frac{N-a^2}{2a} = 9{\cdot}515.$$

The nearest integer to $\sqrt{N-a}$ is in this case b;
the integral part of $\frac{N-a^2}{2a}$ is also b;
but the nearest integer to $\frac{N-a^2}{2a}$ is now $b+1$.

(B.) If the nearest integer to $\sqrt{N-a}$ is $b+1$,
then the nearest integer to $\frac{N-a^2}{2a}$ is also $b+1$;
but now the integral part of $\frac{N-a^2}{2a}$ may be

(1) b ..... [Case B.1],

or (2) $b+1$ ..... [Case B.2],

so that in the case B.2 the ordinary rule is incorrect.

In this proposition $\frac{1}{2} < c < 1$,

therefore $$\tfrac{1}{2} < c + \frac{(b+c)^2}{2a} < \dot{9} + \frac{(10^n - 1 + \dot{9})^2}{2 \cdot 10^{2n}},$$
$$< 1\tfrac{1}{2}.$$

When c is $\frac{1}{2}$, b zero, and n large so that a is large, $c + \frac{(b+c)^2}{2a}$ approaches close to the lower limit. Also it may approach very close to the upper limit $1\frac{1}{2}$.

Consequently the nearest integer to $\frac{N-a^2}{2a}$ is $b+1$ in this case; but

the integral part of $\frac{N-a^2}{2a}$ is sometimes b and sometimes $b+1$.

The mistake which is made in applying the ordinary proof to cases in which the $2n+1$ figures are followed by a decimal, is that this is equivalent to omitting the quantity called c in this paper. Further, on taking account of this quantity c, which can approach very close to unity, it is at once seen that it is not possible to make the rule correct by requiring a to contain more than $n+1$ figures, b containing n as before, for cases can still be found in which $c+\frac{(b+c)^2}{2a}>1$.

Numerical Examples.

Case B.1:

$$N=11992,$$

$$\sqrt{N}=109{\cdot}507\ldots,$$

$$a=100,$$

$$b=9,$$

$$\sqrt{N}-a=9{\cdot}507\ldots,$$

$$\frac{N-a^2}{2a}=9{\cdot}96.$$

The nearest integer to $\sqrt{N}-a$ is in this case $b+1$;
the nearest integer to $\frac{N-a^2}{2a}$ is also $b+1$;
but the integral part of $\frac{N-a^2}{2a}$ is b.

Case B.2:

$$N=11880,$$

$$\sqrt{N}=108{\cdot}9\ldots,$$

$$a=100,$$

$$b=8,$$

$$\sqrt{N}-a=8{\cdot}9\ldots,$$

$$\frac{N-a^2}{2a}=9{\cdot}4.$$

The nearest integer to $\sqrt{N}-a$ is $b+1$ in this case;
the nearest integer to $\frac{N-a^2}{2a}$ is $b+1$;
the integral part of $\frac{N-a^2}{2a}$ is $b+1$;
so that the ordinary rule fails.

Another example will perhaps be a little more striking, as in it the integral part of $\frac{N-a^2}{2a}$ consists of $n+1$ and not n figures.

$$N = 12099,$$

$$\sqrt{N} = 109{\cdot}995\ldots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N}-a = 9{\cdot}995\ldots,$$

$$\frac{N-a^2}{2a} = 10{\cdot}495.$$

The nearest integer to $\sqrt{N}-a$ is $b+1$;
the nearest integer to $\frac{N-a^2}{2a}$ is $b+1$;
the integral part of $\frac{N-a^2}{2a}$ is $b+1$, and consists of two, *i.e.*, $n+1$ figures, for here $n = 1$.

Todhunter gives no general proof that it is possible to extend the rule to cases in which there is a decimal following the $2n+1$ figures. He takes, in effect, the number 12 followed by 8 zeros as an illustration. If, now, the number 12 followed by 3 zeros be taken, then

$$N = 12000,$$

$$\sqrt{N} = 109{\cdot}5\ldots,$$

$$a = 100,$$

$$b = 9,$$

$$\sqrt{N}-a = 9{\cdot}5\ldots,$$

$$\frac{N-a^2}{2a} = 10.$$

So that the rule fails in this case.

Another example of the failure of the rule is

$$N = 14399,$$

$$\sqrt{N} = 119{\cdot}9\ldots,$$

$$a = 110,$$

$$b = 9,$$

$$\sqrt{N-a} = 9{\cdot}9\ldots,$$

$$\frac{N-a^2}{2a} = 10{\cdot}45.$$

The following scheme states concisely the four cases considered above, A.1, A.2, B.1, B.2 :—

Case.	Integral part of $\sqrt{N-a}$.	Nearest integer to $\sqrt{N-a}$.	Integral part of $\frac{N-a^2}{2a}$.	Nearest integer to $\frac{N-a^2}{2a}$.
A.1	b	b	b	b
A.2	b	b	b	$b+1$
B.1	b	$b+1$	b	$b+1$
B.2	b	$b+1$	$b+1$	$b+1$

It shows that the four following questions (one of these questions being answered wrongly in the affirmative by the ordinary rule), which form a symmetrical scheme, must all be answered in the negative :—

1. Is the integral part of $\sqrt{N-a}$ always the same as the integral part of $\frac{N-a^2}{2a}$?

2. Is the nearest integer to $\sqrt{N-a}$ always the same as the integral part of $\frac{N-a^2}{2a}$?

3. Is the integral part of $\sqrt{N-a}$ always the same as the nearest integer to $\frac{N-a^2}{2a}$?

4. Is the nearest integer to $\sqrt{N-a}$ always the same as the nearest integer to $\frac{N-a^2}{2a}$?

Question 1 is answered negatively by Case B.2,

„ 2 „ „ Case B.1,

„ 3 „ „ Cases A.2, B.1, and B.2,

„ 4 „ „ Case A.2.

II. *The rule for contracting the process of finding the Cube Root.*

The rule (see Todhunter's *Algebra*, Art. 257) is this:—

> When $n+2$ figures of a cube root have been obtained by the ordinary method, n more may be obtained by division only, supposing $2n+2$ to be the whole number.

To show that this rule cannot be always extended to cases in which there is a decimal following the $2n+2$ figures, take

$$\sqrt[3]{N} = a+b+c,$$

where a denotes the first $n+2$ figures followed by n zeros, b the remaining n figures, and c the decimal.

Then $$\frac{N-a^3}{3a^2} = b+c+\frac{(b+c)^2}{a}+\frac{(b+c)^3}{3a^2}.$$

In this case, put $$u = c+\frac{(b+c)^2}{a}+\frac{(b+c)^3}{3a^2},$$

so that u is greatest when a is least and b and c are greatest.

The least value of a is 10^{2n+1}.

The greatest value of b is 10^n-1.

The greatest value of c is $\cdot\dot{9}$.

Therefore the greatest value of u is

$$\cdot\dot{9}+\frac{(10^n-1+\cdot\dot{9})^2}{10^{2n+1}}+\frac{(10^n-1+\cdot\dot{9})^3}{3\,.\,10^{4n+2}},$$

which approaches indefinitely close to

$$1+\frac{1}{10}+\frac{1}{3}\,\frac{1}{10^{n+2}}.$$

This is greatest when n is least. Regarding $n=1$ as the least value of n for which the rule is interpretable, the greatest value of u is

$$1+\frac{1}{10}+\frac{1}{3000} = 1{\cdot}100\dot{3}.$$

To construct a case in which the rule fails, put $n = 1$, $a = 1000$, $b = 9$, $c =$ a decimal; then

$$a+b+c < 1010.$$

Take, therefore, $N = (1010)^3 - 1 = 1030300999,$

$$N = 1030300999,$$

$$\sqrt[3]{N} = 1009{\cdot}9\ldots,$$

$$a = 1000,$$

$$b = 9,$$

$$\sqrt[3]{N} - a = 9{\cdot}9\ldots,$$

$$\frac{N-a^3}{3a^2} = 10{\cdot}100333.$$

The ordinary rule would make the integral part of $\frac{N-a^3}{3a^2}$ equal to 9, and not 10 as it really is. $\frac{N-a^3}{3a^2} - b$ in this case is 1·100333, which is less than the maximum excess 1·100$\dot{3}$.

If it be considered possible to interpret the rule for the case $n = 0$, then the cube root contains two figures, which belong wholly to the first part of the root; this can be made as small as possible by putting it $= 10$; in this case b must be considered to vanish, and c will be nearly $= 1$. Take, therefore,

$$N = 11^3 - 1 = 1330,$$

$$N = 1330,$$

$$\sqrt[3]{N} = 10{\cdot}9\ldots,$$

$$a = 10,$$

$$b = 0$$

[b is considered to vanish if rule be interpretable],

$$\sqrt[3]{N} - a = 0{\cdot}9\ldots,$$

$$\frac{N-a^3}{3a^2} = 1{\cdot}1.$$

The integral part of $\sqrt[3]{N} - a$ is 0, but that of $\frac{N-a^3}{3a^2}$ is 1; and the rule, if interpretable, fails.

February 10*th*, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. J. R. Holt, B.A., T.C.D., J.P. Banda, India, was elected a Member.

The following communications were made :—

On the Equation of Riccati: the President (Prof. Hart taking the chair *pro tem.*).

The "Orthocentroidal" Triangle: R. Tucker.

On Polygons inscribed in a Quadric and circumscribed about two Confocal Quadrics: R. A. Roberts.

On the Binomial Equation $x^p - 1 = 0$, Quinquisection: Prof. Lloyd Tanner.

Symmetrical Determinant Formulæ in Elliptic Functions: L. J. Rogers.

A Note on Curves: H. M. Taylor.

Some Generalisations of Formulæ connected with the change of the Independent Variable in a Differential Expression, with application to a new class of Reciprocants: C. Leudesdorf.

On the initiative of Prof. Sylvester, the wish was strongly expressed by the members present that, either authors should endeavour to be present to make their communications *vivâ voce*, or, failing their ability to be present, should send concise but sufficiently full abstracts of their papers, so as to enable the members to appreciate the gist of the communications.

The following presents were received :—

"Proceedings of the Royal Society," Vol. XLI., Nos. 248 and 249.

"Educational Times," for February.

"Proceedings of the Cambridge Philosophical Society," Vol. V., Part VI.; Camb., 1886.

"Bulletin des Sciences Mathématiques," Tome XI., Jan., 1887.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XI., St. 1.

"Bollettino delle Pubblicazioni Italiane," ricevute per diritto di Stampa, Nos. 25 and 26.

"Sitzungsberichte der Physikalisch-medizinischen Societät zu Erlangen," 18 Heft, 1886.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 12; Roma, 1886.

On the Equation of Riccati. *By* Sir James Cockle, M.A., F.R.S.

[*Read Feb.* 10*th*, 1887.]

1. This paper will consist of seven sections :—

I. On Deformations, Mixed Integrals, Characteristics, and Correlations.
II. On Synthemes.
III. On Terordinals.
IV. On Characteristics of Terordinals.
V. On Transformation to Riccatian Forms.
VI. On a Particular Terordinal.
VII. Conclusion.

I. *On Deformations, Mixed Integrals, Characteristics, and Correlations.*

2. Meaning by f, F, ϕ, and Φ, respectively, the four expressions

$$f\left(y, x, \frac{d}{dx}\right), \qquad F\left(Y, x, \frac{d}{dx}\right),$$

$$\phi\left(y, Y, x, \frac{d}{dx}\right), \quad \Phi\left(Y, y, x, \frac{d}{dx}\right),$$

and denoting by acute accents differential coefficients taken with respect to the independent variable x, suppose that

$$\Phi f + \phi F = \frac{d\Omega}{dx} = \Omega';$$

then I call either of the functions f and F a deformation of the other, and $\Omega - c$ a mixed integral of the system f and F.

3. The presence of Y is essential to Φ, and that of y to ϕ; but in particular cases y may be absent from Φ, and Y from ϕ.

4. Let each of the values u, v, w ... of y reduce f to zero, and let ϕ_u, ϕ_v, ϕ_w ... be the corresponding values of ϕ. Then ϕ_u, ϕ_v, ϕ_w ... are integrating factors of F. So, if each of the values U, V, W ... of Y reduce F to zero, the corresponding values of Φ, say Φ_U, Φ_V, Φ_W ... are integrating factors of f.

5. When f and F are of the same order, the complete solution of one involves that of the other.

6. Any general relation of the form

$$\chi\left(y, Y, x, \frac{d}{dx}\right) = 0$$

I call a characteristic. By eliminating Y between the characteristic and the mixed integral, we get a first integral of f, and by so eliminating y we get a first integral of F. Such integral contains an arbitrary constant.

7. In f substitute for x a function of another variable, say t, and after the substitution replace t by x. We thus get a new expression which I shall represent by $\dot{f}\left(\dot{y}, x, \frac{d}{dx}\right)$. An accompanying deformation, corresponding with F, I shall denote by $\dot{F}\left(\dot{Y}, x, \frac{d}{dx}\right)$; calling the passages from f to $\dot{f}$, and from F to $\dot{F}$ correlations, and f, $\dot{f}$ or F, $\dot{F}$ correlates.

II. *On Synthemes.*

8. Denoting r accents by the index (r), I call such an expression as $\dot{u}^{(r)}u^{(s)}+\dot{v}^{(r)}v^{(s)}$, which may be expressed briefly by $S.\dot{u}^{(r)}u^{(s)}$, or by (r, s), a syntheme. Put

$$\dot{u}u+\dot{v}v = A, \quad \dot{u}'u+\dot{v}'v = B, \quad \dot{u}u'+\dot{v}v' = C,$$
$$\dot{u}'u'+\dot{v}'v' = D, \quad \dot{u}''u+\dot{v}''v = E, \quad \dot{u}u''+\dot{v}v'' = F,$$
$$\dot{u}''u''+\dot{v}''v'' = I, \quad \dot{u}''u'+\dot{v}''v' = G, \quad \dot{u}'u''+\dot{v}'v'' = H,$$

and we have nine relations between the thirteen quantities $A, B, \ldots I$, $\dot{u}$, u, $\dot{v}$ and v.

9. On inspection, we see that

$$A' = B+C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$
$$B' = D+E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$
$$C' = D+F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$
$$D' = G+H \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d),$$

and, by the elimination of $\dot{v}$ and v, we get

$$AD-BC = A\dot{u}'u' - B\dot{u}u' - C\dot{u}'u + D\dot{u}u \quad \ldots\ldots\ldots\ldots(1),$$

$$AI-EF = A\dot{u}''u'' - E\dot{u}u'' - F\dot{u}''u + I\dot{u}u \quad \ldots\ldots\ldots\ldots(2),$$

$$DI-GH = D\dot{u}''u'' - G\dot{u}'u'' - H\dot{u}''u' + I\dot{u}'u' \quad \ldots\ldots\ldots\ldots(3),$$

and we have seven relations between the eleven quantities A, B, ... I, $\dot{u}$ and u.

10. There is also the relation, obtainable by elimination,

$$A(G+H)-CE-BF$$

$$= A(\dot{u}''u'+\dot{u}'u'')-B\dot{u}u''-C\dot{u}''u-E\dot{u}u'-F\dot{u}'u+(G+H)\dot{u}u \ldots(4),$$

but this last relation is not independent, being deducible by differentiation from (1).

11. Assume $y'''+py''+qy'+ry=0 \quad \ldots\ldots\ldots\ldots(\text{i.}),$

$$\dot{y}'''+\dot{p}\dot{y}''+\dot{q}\dot{y}'+\dot{r}\dot{y}=0 \quad \ldots\ldots\ldots\ldots(\text{ii.}).$$

12. Then, recalling Art. 8 and its notation,

$$E' = G+\dot{u}'''u+\dot{v}'''v = G+(3,0) = G-\dot{p}E-\dot{q}B-\dot{r}A \ldots\ldots\ldots\ldots(e),$$

$$F' = H+(0,3) = H-pF-qC-rA \ldots\ldots\ldots\ldots(f),$$

$$G' = I+(3,1) = I-\dot{p}G-\dot{q}D-\dot{r}C \ldots\ldots\ldots\ldots(g),$$

$$H' = I+(1,3) = I-pH-qD-rB \ldots\ldots\ldots\ldots(h),$$

$$I' = (3,2)+(2,3) = -(\dot{p}+p)I-\dot{q}H-\dot{r}F-qG-rE$$

$$\ldots\ldots\ldots\ldots(i),$$

but these last relations (e), (f), ... (i) are not independent, although, like (4), they subserve the purposes of elimination.

13. The nine independent relations, in Arts. 9 and 11, between the eleven quantities leave the system indeterminate. But the system, being homogeneous, will be rendered determinate when one more relation is assigned.

III. *On Terordinals.*

14. Assume $Y'''+PY''+QY'+RY=0 \quad \ldots\ldots\ldots\ldots(\text{iii.}),$

and that between the coefficients of (i.) and (iii.), or, recurring to

Art. 1, say of $f = 0$ and $F = 0$, there subsist the three (two skew and one normal) symmetrical relations

$$R = -r,$$

$$P + \frac{R'}{R} = -\left(p + \frac{r'}{r}\right),$$

$$Q - \tfrac{1}{2}\left(P + \frac{R'}{R}\right) + \tfrac{1}{2}\frac{R'}{R}\left(P + \frac{R'}{R}\right) = q - \tfrac{1}{2}\left(p + \frac{r'}{r}\right)' + \tfrac{1}{2}\frac{r'}{r}\left(p + \frac{r'}{r}\right),$$

then (iii.) is a deformation of (i.), and (i.) of (iii.).

15. For if $\Phi = \dfrac{Y'}{R}$ and $\phi = \dfrac{y'}{r}$, then

$$\Omega' = \frac{d}{dx}\left\{\frac{1}{r}(y'Y'' - y''Y') - \frac{1}{r}\left(p + \frac{r'}{r}\right)y'Y' - yY\right\};$$

whence the mixed integral

$$\Omega - c_0 = \frac{1}{r}(y'Y'' - y''Y') - \frac{1}{r}\left(p + \frac{r'}{r}\right)y'Y' - yY.$$

16. If f and F be supposed to vanish, we may replace Ω by an arbitrary constant, and thus get

$$y'Y'' - y''Y' - \left(p + \frac{r'}{r}\right)y'Y' - ryY = cr.$$

17. Let u, v and w be independent particular solutions of (i.), and U, V, W of (iii.). Then, by means of the mixed integral, we get the systems of values exhibited by means of determinants in the following eighteen equations.

18. For one system, we have, putting

$$p + \frac{r'}{r} = \pi,$$

$$U = e^{\int p\,dx}\begin{vmatrix} v'', & v' \\ w'', & w' \end{vmatrix}, \quad V = e^{\int p\,dx}\begin{vmatrix} w'', & w' \\ u'', & u' \end{vmatrix}, \quad W = e^{\int p\,dx}\begin{vmatrix} u'', & u' \\ v'', & v' \end{vmatrix},$$

$$U' = re^{\int p\,dx}\begin{vmatrix} v', & v \\ u', & u \end{vmatrix}, \quad V' = re^{\int p\,dx}\begin{vmatrix} w', & w \\ u', & u \end{vmatrix}, \quad W' = re^{\int p\,dx}\begin{vmatrix} u', & u \\ v', & v \end{vmatrix},$$

$$U'' = \pi U' + re^{\int p\,dx} \begin{vmatrix} v'', & v \\ w'', & w \end{vmatrix}, \quad V'' = \pi V' + re^{\int p\,dx} \begin{vmatrix} w'', & w \\ u'', & u \end{vmatrix},$$

$$W'' = \pi W' + re^{\int p\,dx} \begin{vmatrix} u'', & u \\ v'', & v \end{vmatrix}.$$

19. For another, we have, putting

$$P + \frac{R'}{R} = \Pi,$$

$$u = e^{\int P\,dx} \begin{vmatrix} V'', & V' \\ W'', & W' \end{vmatrix}, \quad v = e^{\int P\,dx} \begin{vmatrix} W'', & W' \\ U'', & U' \end{vmatrix}, \quad w = e^{\int P\,dx} \begin{vmatrix} U'', & U' \\ V'', & V' \end{vmatrix},$$

$$u' = Re^{\int P\,dx} \begin{vmatrix} V', & V \\ W', & W \end{vmatrix}, \quad v' = Re^{\int P\,dx} \begin{vmatrix} W', & W \\ U', & U \end{vmatrix}, \quad w' = Re^{\int P\,dx} \begin{vmatrix} U', & U \\ V', & V \end{vmatrix},$$

$$u'' = \Pi u' + Re^{\int P\,dx} \begin{vmatrix} V'', & V \\ W'', & W \end{vmatrix}, \quad v'' = \Pi v' + Re^{\int P\,dx} \begin{vmatrix} W'', & W \\ U'', & U \end{vmatrix},$$

$$w'' = \Pi w + Re^{\int P\,dx} \begin{vmatrix} U'', & U \\ V'', & V \end{vmatrix}.$$

20. In verification put, for the moment,

$$\phi = re^{\int p\,dx} \begin{vmatrix} v'', & v \\ w'', & w \end{vmatrix},$$

then $$U'' = \pi U' + \phi, \quad \text{or} \quad U'' - \left(p + \frac{r'}{r}\right) U' = \phi;$$

whence $$U''' - \left(p + \frac{r'}{r}\right) U'' - \left(p + \frac{r'}{r}\right)' U' = \phi';$$

but $$\phi' = \left(p + \frac{r'}{r}\right) \phi + rU + re^{\int p\,dx} \begin{vmatrix} v''', & v \\ w''', & w \end{vmatrix},$$

and $$\begin{vmatrix} v''', & v \\ w''', & w \end{vmatrix} = \begin{vmatrix} -pv'' - qv' - rv, & v \\ -pw'' - qw' - rw, & w \end{vmatrix} = -p \begin{vmatrix} v'', & v \\ w'', & w \end{vmatrix} - q \begin{vmatrix} v', & v \\ w', & w \end{vmatrix};$$

therefore $$\phi' = \frac{r'}{r}\phi + rU - qU' = \frac{r'}{r}(U'' - \pi U') + rU - qU',$$

and $U'''-\left(p+2\frac{r'}{r}\right)U''+\left\{q-\left(p+\frac{r'}{r}\right)'+\frac{r'}{r}\left(p+\frac{r'}{r}\right)\right\}U'-rU=0$

.....(iii. *bis*),

which is, in fact, (iii.).

21. Mere differentiations will give one class of verifications. Another will be obtained by substitutions. Thus,

$$\begin{aligned} w &= e^{\int P\,dx}(V'U''-U'V'') \\ &= e^{\int P\,dx}\cdot re^{\int p\,dx}\{V'(wv''-vw'')-U'(uw''-wu'')\} \\ &= r^2e^{\int(P+2p)\,dx}\{(uw'-wu')(wv''-vw'')-(wv'-vw')(uw''-wu'')\} \\ &= e^{\int p\,dx}\cdot w\cdot\begin{vmatrix} u'', & u', & u \\ v'', & v', & v \\ w'', & w', & w \end{vmatrix} = w \text{ (as it ought)};\end{aligned}$$

for, denoting the last determinant by δ, we have, neglecting vanishing determinants,

$$\delta'=\begin{vmatrix} u''', & u', & u \\ v''', & v', & v \\ w''', & w', & w \end{vmatrix}=\begin{vmatrix} -pu''-qu'-ru, & u', & u \\ -pv''-qv'-rv, & v', & v \\ -pw''-qw'-rw, & w', & w \end{vmatrix}=-p\delta;$$

and $$\delta=e^{-\int p\,dx} \quad \text{or} \quad e^{\int p\,dx}\cdot\delta=1.$$

22. So, when $$\Delta=\begin{vmatrix} U'', & U', & U \\ V'', & V', & V \\ W'', & W', & W \end{vmatrix},$$

then also $$\Delta=e^{-\int P\,dx} \quad \text{or} \quad e^{\int P\,dx}\cdot\Delta=1.$$

23. An arbitrary element is involved in δ, for a multiple of a solution is likewise a solution. The same is true of Δ. Considered isolatedly, every one of the expressions δ, Δ, $e^{\int p\,dx}$ and $e^{\int P\,dx}$ carries with it an independent arbitrary multiplier.

24. A third class of verifications may be got from the ternary quadratic synthemes

$$u^{(r)}U^{(s)}+v^{(r)}V^{(s)}+w^{(r)}W^{(s)}, \text{ or, say, } S.u^{(r)}U^{(s)}.$$

The relations $S.u^{(r)}U^{(s)} = S.U^{(s)}u^{(r)}$

are, or ought to be, identical equalities. Thus,

$$S.uU = e^{\int p\,dx}.\delta = 1 = e^{\int P\,dx}.\Delta = S.Uu.$$

25. Take another identity, say

$$S.u''U' = S.U'u''.$$

Here $S.u''U' = r,$

and $S.U'u'' = \Pi S.U'u'$

$$+Re^{\int P\,dx}\{U''(W'V-V'W)+V''(U'W-W'U)+W''(V'U-U'V)\};$$

but $S.U'u' = 0 = S.u'U'$ and $S.U'u'' = -R = r,$

as it ought.

Therefore, further,

$$S.u''U'' = \pi r \quad \text{and} \quad S.U''u'' = \Pi R = (-\pi)(-r) = \pi r.$$

IV. *On Characteristics of Terordinals.*

26. When in (i.), we put

$$e^{\frac{1}{3}\int p\,dx}y = z,$$

we get $z''' + gz' + hz = 0;$

wherein $g = q - \frac{1}{3}p^2 - p',$

and $h = r - \frac{1}{3}pq + \frac{2}{27}p^3 - \frac{1}{3}p''.$

27. So, when in (iii.) we put

$$e^{\frac{1}{3}\int P\,dx}Y = Z,$$

we get $Z''' + GZ' + HZ = 0;$

wherein $G = Q - \frac{1}{3}P^2 - P',$

and $H = R - \frac{1}{3}PQ + \frac{2}{27}P^3 - \frac{1}{3}P''.$

28. Suppose that $\frac{Y}{y}$ = some function, say X, of x, and that what-

ever (value of) Y be taken a value of y can be found satisfying $Y = Xy$. Then the equations in z and Z will be in fact the same equation, and both $g = G$ and $h = H$ will be identically true. Substituting for P, Q, and R, we get from $g = G$ the condition

$$q - \tfrac{1}{3}p^2 - p' = q + \left(p + \frac{r'}{r}\right)\frac{r'}{r} - \left(p + \frac{r'}{r}\right)' - \tfrac{1}{3}\left(p + 2\frac{r'}{r}\right)^2 + \left(p + 2\frac{r'}{r}\right)',$$

which reduces to $$\left(p + \frac{r'}{r}\right)' - \tfrac{1}{3}\left(p + \frac{r'}{r}\right)\frac{r'}{r} = 0 \quad \text{.....(I.),}$$

whence $$p = kr^3 - \frac{r'}{r} \quad \text{.....(II.),}$$

wherein k is an arbitrary constant introduced by integration.

29. From $h = H$, we get

$$r - \tfrac{1}{3}pq + \tfrac{2}{27}p^3 - \tfrac{1}{3}p'' = -r + \tfrac{1}{3}\left(p + 2\frac{r'}{r}\right)\left\{q + \left(p + \frac{r'}{r}\right)\frac{r'}{r} - \left(p + \frac{r'}{r}\right)'\right\}$$
$$- \tfrac{2}{27}\left(p + 2\frac{r'}{r}\right)^3 + \tfrac{1}{3}\left(p + 2\frac{r'}{r}\right)'',$$

or, remembering (I.), and transposing,

$$2r = \tfrac{2}{3}\left(p + \frac{r'}{r}\right)q + \tfrac{2}{9}\left(p + 2\frac{r'}{r}\right)\left(p + \frac{r'}{r}\right)\frac{r'}{r}$$
$$- \tfrac{2}{27}\left\{p^3 + \left(p + 2\frac{r'}{r}\right)^3\right\} + \tfrac{2}{3}\left(p + \frac{r'}{r}\right).$$

30. But

$$p^3 + \left(p + 2\frac{r'}{r}\right)^3 = 2\left(p + \frac{r'}{r}\right)\left\{\left(p + \frac{r'}{r}\right)^2 + 3\left(\frac{r}{r}\right)^2\right\},$$

and, by (I.),

$$\left(p + \frac{r'}{r}\right)'' = \tfrac{1}{9}\left(p + \frac{r'}{r}\right)\left(\frac{r'}{r}\right)^2 + \tfrac{1}{3}\left(p + \frac{r'}{r}\right)\left(\frac{r'}{r}\right)'.$$

31. Replacing $p + \frac{r'}{r}$ by its value, kr^3, obtained from (II.), we shall find

$$q = \left(\frac{2k^2}{9} + \frac{3}{k}\right)r^3 - \frac{k}{3}r^{-3}r' + \tfrac{2}{9}\left(\frac{r'}{r}\right)^2 - \tfrac{1}{3}\left(\frac{r'}{r}\right) \quad \text{.....(III.).}$$

32. And, since

$$Q = q + \left(p + \frac{r'}{r}\right)\frac{r'}{r} - \left(p + \frac{r'}{r}\right) = q + kr^{\frac{3}{2}}\frac{r'}{r} - (kr^{\frac{3}{2}})' = q + \tfrac{2}{3}kr^{-\frac{3}{2}}r',$$

we have

$$P = -kr^{\frac{3}{2}} - \frac{r'}{r},$$

and

$$Q = \left(\frac{2k^2}{9} + \frac{3}{k}\right)r^3 + \frac{k}{3}r^{-\frac{3}{2}}r' + \tfrac{2}{9}\left(\frac{r'}{r}\right)^2 - \tfrac{1}{3}\left(\frac{r'}{r}\right)',$$

so that P, Q, and R are respectively derived from p, q, and r by simply changing the sign of r. The only term in q affected by the change is that containing r'.

33. We might (see Art. 8) anticipate that, with the values of p and q given by (II.) and (III.), we should be able to get a first integral of (i.), as in fact we can. For $e^{\frac{3}{2}\int p\,dx}y$ is an integrating factor, and the integral is

$$yy'' - \tfrac{1}{2}y'^2 + \tfrac{1}{3}pyy' + \frac{3}{2k}r^{\frac{3}{2}}y^2 = ce^{-\frac{3}{2}\int p\,dx},$$

wherein c is the arbitrary constant.

34. But, with these particular values of p and q, we can completely integrate (i.), which admits of the symbolical decompositions

$$\left(\frac{d}{dx} + \tfrac{1}{3}kr^{\frac{3}{2}} - \tfrac{2}{3}\frac{r'}{r}\right)\left\{\frac{d^2}{dx^2} + \left(\tfrac{2}{3}kr^{\frac{3}{2}} - \tfrac{1}{3}\frac{r'}{r}\right)\frac{d}{dx} + \frac{3}{k}r^{\frac{3}{2}}\right\}y = 0,$$

and

$$\left(\frac{d}{dx} + \tfrac{1}{3}kr^{\frac{3}{2}} - \tfrac{2}{3}\frac{r'}{r}\right)\left\{\frac{d}{dx} + (\tfrac{1}{3}k \pm K)r^{\frac{3}{2}} - \tfrac{1}{3}\frac{r'}{r}\right\}\left\{\frac{d}{dx} + (\tfrac{1}{3}k \mp K)r^{\frac{3}{2}}\right\}y = 0,$$

where

$$K = \left(\tfrac{1}{9}k^2 - \frac{3}{k}\right)^{\frac{1}{2}};$$

and which is transformed into an equation with constant coefficients by changing the independent variable from x to t if t be determined from

$$Ct = \int r^{\frac{3}{2}}dx,$$

C being any constant.

35. Thus the simplest form of characteristic happens to indicate a completely integrable equation. But it also verifies the form of the

mixed equation. For z and Z are identical, and

$$X = \frac{Y}{y} = e^{\frac{1}{3}\int (p-P)\,dx} = e^{\frac{2}{3}\int \pi\,dx}.$$

Changing signs in the mixed integral of Art. 16, and substituting therein $e^{\frac{1}{3}\int \pi\,dx} y$ for Y, we get, after developments and reductions,

$$yy'' - \tfrac{1}{2}y'^2 + \left(\tfrac{1}{3}\pi - \frac{\pi'}{\pi}\right) yy' + \tfrac{3}{2}\frac{r}{\pi}y^2 = -\tfrac{3}{2}\frac{cr}{\pi}e^{-\frac{2}{3}\int \pi\,dx};$$

but $\pi = kr^{\frac{1}{3}}$, and this integral and that of Art. 33 agree.

36. In verifying this integral by differentiation, it will be noticed that

$$q = \tfrac{2}{9}p^2 + \tfrac{1}{3}p' + \frac{3}{k}r^{\frac{1}{3}} \text{ and } r = \frac{1}{k}(r^{-\frac{1}{3}}r' + pr^{\frac{2}{3}}),$$

and also that the mixed integral gives the integrating factor.

V. *On Transformation to Riccatian Forms.*

37. Used in connection with the processes of Boole by D, I mean, of course, $x\frac{d}{dx}$. Let

$$D(D-2B)\,y - 2\lambda(D-2A)\,x^2y = 0 \quad \ldots\ldots\ldots\ldots\ldots [1],$$

where, for the present purpose, A and B denote given constants. Change the independent variable from x to t $(= x^2)$, and then write x instead of t. We get

$$D(D-B)\,y - \lambda(D-A)\,xy = 0.$$

38. Put $y = e^x z$; then

$$Dy = e^x(Dz + xz) \text{ and } D^2y = e^x\{D^2z + 2D(xz) - xz + x^2z\},$$

for

$$xDz = D(xz) - xz.$$

Hence $D(D-B)\,z + (2-\lambda)\,D(xz) + (\lambda A - B - 1)\,xz + x^2z = 0$;

and, if $\lambda = 2$ and $2A - B - 1 = 0$, then

$$D(D-B)\,z + x^2z = 0,$$

which is equivalent to

$$\frac{d^2z}{dx^2} - \frac{B-1}{x}\frac{dz}{dx} + z = 0,$$

which last may be transformed into

$$\frac{d^2z}{dx^2}+x^{2q-2}z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots[2],$$

as appears from Mr. J. W. L. Glaisher's standard paper (*Phil. Trans.*, 1881).

39. By Boole's method we can change A into $A+i$ and B into $B+j$, where i and j are any integers. So that we may enlarge

$$2A-B-1=0 \text{ into } 2(A+i)-(B+j)-1=0.$$

But if $2A$ be an uneven integer, say $2k+1$, then $B+j\,(=2k+2i)$ is an even integer, and $q\left(=\frac{1}{B+j}\right)$ is the reciprocal of an even integer.

40. The case hereinafter discussed is that in which $2A=1$ and $B=1$, and for which $2A-B-1$ is not satisfied, but which can be transformed into cases of

$$D\{D-2(B+j)\}y-2\lambda\{D-2(A+i)\}x^2y=0,$$

wherein the condition is satisfied. Here we transform to

$$D\{D-2(1+j)\}y-2\lambda\{D-(1+2i)\}x^2y=0,$$

with the condition $j=2i-1$; viz., to

$$D(D-4i)y-2\lambda\{D-(1+2i)\}x^2y=0,$$

say to $$D(D-2b)-2\lambda(D-2a)x^2y=0.$$

Here $b=2i$, $2a=2i+1$, and $2a-b-1=0$, so that the requisite condition is satisfied. The only exceptional case is when $i=0$.

VI. *On a Particular Terordinal.*

41. Take the terordinal

$$y'''+\left(\frac{1}{x}+x\right)y''-\left(\frac{1}{x^2}+2\right)y'+\frac{2}{x}y=0 \quad \ldots\ldots\ldots\ldots(\text{i.}),$$

or which $p+\frac{r'}{r}=\pi=x$, $\pi\frac{r'}{r}=-1$ and $\pi'=1$.

42. The deformation of (i.) is therefore

$$Y'''+\left(\frac{1}{x}-x\right)Y''-\left(\frac{1}{x^2}+4\right)Y'-\frac{2}{x}Y=0 \quad \ldots\ldots\ldots(\text{iii.}).$$

43. Form a correlate by changing x into $t\sqrt{-1}$, and then replacing t by x. The result is

$$\dot{y}''' + \left(\frac{1}{x} - x\right)\dot{y}'' - \left(\frac{1}{x^2} - 2\right)\dot{y}' - \frac{2}{x}\dot{y} = 0 \quad \text{.....(ii.),}$$

for which $\dot{p} + \frac{\dot{r}}{r} = \dot{\pi} = -x$, $\dot{\pi}\,\frac{\dot{r}'}{r} = 1$, and $\dot{\pi} = -1$.

44. The deformation of (ii.) is therefore

$$\dot{Y}''' + \left(\frac{1}{x} + x\right)\dot{Y}'' - \left(\frac{1}{x^2} - 4\right)\dot{Y}' + \frac{2}{x}\dot{Y} = 0 \quad \text{.....(iv.).}$$

45. Expressed in the notation of Boole, we should have the given equation $D^2(D-2)\,y + (D-3)(D-4)\,x^2 y = 0$,

its deformation $D^2(D-2)\,Y - D(D-1)\,x^2 Y = 0$,

a correlate $D^2(D-2)\,\dot{y} - (D-3)(D-4)\,x^2\dot{y} = 0$,

and its deformation $D^2(D-2)\,\dot{Y} + D(D-1)\,x^2\dot{Y} = 0$.

46. Multiplied into $(D-1)(D-2)$, the given equation becomes, after reductions,

$$\left\{D(D-2) + (D-1)\,x^2\right\}[D]^3 y = 0,$$

wherein the $[D]^3$ is factorial.

47. Integrate the deformation of the correlate once, and make the arbitrary constant zero; then

$$\left\{D(D-2) + (D-1)\,x^2\right\}(\dot{\lambda}\dot{Y}) = 0,$$

wherein $\dot{\lambda}$ is an arbitrary constant multiplier.

48. Comparing the last two equations, we get

$$\dot{\lambda}\dot{Y} = [D]^3 y = x^3 y'''.$$

49. So, from the correlate and the deformation of the given equation, we get

$$\lambda Y = [D]^3\dot{y} = x^3\dot{y}'''.$$

50. The last two relations hold when general values of y and $\dot{y}$ are

inserted; for $\dot{y} = x^2$ and $y = x^2$ are respectively solutions of the given equation and of the correlate; and $[D]^3 x^2 = 0$.

51. The three equations

$$D(D-2)\,Y-(D-1)\,x^2 Y = 0, \quad x^2 Y''-(x+x^3)\,Y'-x^2 Y = 0,$$

$$Y''-\left(\frac{1}{x}+x\right)Y''-Y = 0,$$

are in substance the same. So are the corresponding three

$$D(D-2)\,\dot{Y}+(D-1)\,x^2\dot{Y} = 0, \quad x^2\dot{Y}''-(x-x^3)\,\dot{Y}'+x^2\dot{Y} = 0,$$

$$\dot{Y}''-\left(\frac{1}{x}-x\right)\dot{Y}+\dot{Y} = 0.$$

52. Taking u, v and w as independent particular solutions of (i.), and making $w = x^2$, we get, from the formulæ of § III.,

$$U = e^{\int p\,dx}(w'v''-v'w'') = 2e^{\int p\,dx}(xv''-v') = 2e^{\int p\,dx}(xv'-2v)$$

$$V = e^{\int p\,dx}(u'w''-w'u'') = 2e^{\int p\,dx}(u'-xu'') = -2e^{\int p\,dx}(xu'-2u)',$$

$$x^2\,(=w) = e^{\int P\,dx}(V'U''-U'V''),$$

which (see Art. 51)

$$= e^{\int P\,dx}\left[V'\left\{\left(\frac{1}{x}+x\right)U'+U\right\}-U'\left\{\left(\frac{1}{x}+x\right)V'+V\right\}\right]$$

$$= e^{\int P\,dx}(UV'-VU').$$

53. Differentiating, we get

$$U' = pU+2e^{\int p\,dx}(xv''') = pU+\frac{2\dot{\lambda}}{x^2}\dot{V}.\,e^{\int p\,dx},$$

$$V' = pV-2e^{\int p\,dx}(xu''') = pV-\frac{2\dot{\lambda}}{x^2}\dot{U}.\,e^{\int p\,dx},$$

therefore $$UV'-VU' = -\frac{2\dot{\lambda}}{x^2}(\dot{U}U+\dot{V}V)\,e^{\int p\,dx},$$

and $$x^2 = -\frac{2\dot{\lambda}}{x^2}e^{\int (p+P)\,dx}(\dot{U}U+\dot{V}V).$$

54. Noticing that $\dot{p} = P$, $\dot{P} = p$, and that we may (being always able to employ arbitrary multipliers) put $\dot{w} = x^2 = w$, we shall, by

processes corresponding precisely with those of Arts. 52 and 53, find

$$x^2 = +\frac{2\lambda}{x^2} e^{\int (P+p)\,dx} (U\dot{U}+V\dot{V}),$$

the + sign being prefixed to this result because the last line of Art. 51 differs from the second. Hence, reducing, and equating the values of x^2,

$$-2\dot{\lambda}\,(\dot{U}U+\dot{V}V) = x^2 = 2\lambda\,(U\dot{U}+V\dot{V}),$$

whence

$$\dot{\lambda}+\lambda = 0.$$

55. The process indicated in Art. 4 might have been used to obtain the integrals given in Art. 51. But its application is not necessary, for Boole's notation renders the existence of the integrals clear.

56. For brevity, put

$$xu'-2u = \theta, \quad xv'-2v = \vartheta,$$

$$x\dot{u}'-2\dot{u} = \dot{\theta}, \quad x\dot{v}'-2\dot{v} = \dot{\vartheta};$$

then

$$U = 2e^{\int p\,dx}\,\vartheta', \quad V = -2e^{\int p\,dx}\,\theta',$$

$$\dot{U} = 2e^{\int P\,dx}\,\dot{\vartheta}' \text{ and } \dot{V} = -2e^{\int P\,dx}\,\dot{\theta}'.$$

57. Hence

$$\dot{U}U+\dot{V}V = 4e^{\int (p+P)\,dx}\,(\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta') = 4mx^2\,(\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta'),$$

and

$$\begin{aligned}\dot{\theta}'\theta'+\dot{\vartheta}'\vartheta' &= (x\dot{u}''-\dot{u}')(xu''-u')+(x\dot{v}''-\dot{v}')(xv''-v') \\ &= x^2\,(\dot{u}''u''+\dot{v}''v'')-x\,(\dot{u}''u'+\dot{v}''v') \\ &\qquad -x\,(\dot{u}'u''+\dot{v}'v'')+\dot{u}'u'+\dot{v}'v' \\ &= x^2 I-xG-xH+D, \text{ (say } = \Theta_3),\end{aligned}$$

when expressed by the notation used for Synthemes; therefore

$$x^2 I-xG-xH+D = \Theta_3 = \frac{-1}{8m\dot{\lambda}} = \frac{1}{8m\lambda} = M,$$

where M is an arbitrary constant meaning the same thing as Θ_3.

58. By this mode of proceeding we, in short, obtain the groups of

four relations,

$$\dot\theta\theta + \dot\vartheta\vartheta = x^2D - 2xB - 2xC + 4A, \quad \text{say} = \Theta_0,$$
$$\theta'\theta + \dot\vartheta'\vartheta = x^2G - xD - 2xE + 2B, \quad = \Theta_1,$$
$$\dot\theta\theta' + \dot\vartheta\vartheta' = x^2H - xD - 2xF + 2C, \quad = \Theta_2,$$
$$\dot\theta'\theta' + \dot\vartheta'\vartheta' = x^2I - xG - xH + D, \quad = \Theta_3.$$

59. We also have

$$\dot\lambda\dot U = x^3u''' = -x^2(1+x^2)u'' + x(1+2x^2)u' - 2x^2u$$
$$= -x(1+x^2)(xu''-u') + x^2(xu'-2u) = -x(1+x^2)\theta' + x^2\theta,$$

and, in short, the groups of four relations,

$$\dot\lambda\dot U = -x(1+x^2)\theta' + x^2\theta = 2\dot\lambda e^{\int P\,dx}\dot\vartheta' \quad \ldots\ldots\ldots\ldots\{1\},$$
$$\dot\lambda\dot V = -x(1+x^2)\vartheta' + x^2\vartheta = -2\dot\lambda e^{\int P\,dx}\dot\theta' \quad \ldots\ldots\ldots\ldots\{2\},$$
$$\lambda U = -x(1-x^2)\dot\theta' - x^2\dot\theta = 2\lambda e^{\int p\,dx}\vartheta' \quad \ldots\ldots\ldots\ldots\{3\},$$
$$\lambda V = -x(1-x^2)\dot\vartheta' - x^2\dot\vartheta = -2\lambda e^{\int p\,dx}\theta' \ldots\ldots\ldots\ldots\{4\}.$$

60. Hence

$$\lambda\dot\lambda(U\dot U + V\dot V) = x^2(1-x^4)(\theta'\dot\theta' + \vartheta'\dot\vartheta') + x^3(1+x^2)(\theta'\dot\theta + \vartheta'\dot\vartheta)$$
$$-x^3(1-x^3)(\theta\dot\theta' + \vartheta\dot\vartheta') - x^4(\theta\dot\theta + \vartheta\dot\vartheta)$$
$$= 4\lambda\dot\lambda e^{\int(p+P)\,dx}(\theta'\dot\theta' + \vartheta'\dot\vartheta') = 4m\lambda\dot\lambda x^2(\theta'\dot\theta' + \vartheta'\dot\vartheta');$$

whence, in the notation of Synthemes,

$$(1-4m\lambda\dot\lambda - x^4)\Theta_3 + x(1+x^2)\Theta_2 - x(1-x^2)\Theta_1 - x^2\Theta_0 = 0.$$

61. The value of $U\dot U + V\dot V$ just indicated ought, and will be shown, to agree, in the results which it gives, with that of Art. 57.

62. Replacing, in Art. 59, $e^{\int p\,dx}$ by $kxe^{\frac{1}{2}x^2}$ and $e^{\int P\,dx}$ by $\dot kxe^{-\frac{1}{2}x^2}$, an x will divide out from the second and third columns. Form the four equations,

$$2k\lambda e^{\frac{1}{2}x^2}\{1\} + (1+x^2)\{4\}, \qquad -2k\lambda e^{\frac{1}{2}x^2}\{2\} + (1+x^2)\{3\},$$
$$2\dot k\dot\lambda e^{-\frac{1}{2}x^2}\{4\} - (1-x^2)\{1\}, \qquad -2\dot k\dot\lambda e^{-\frac{1}{2}x^2}\{3\} - (1-x^2)\{2\},$$

from which θ', ϑ', $\dot\vartheta'$ and $\dot\theta'$ are, respectively, absent, and we get a system of four equations,

$$(x^4-1-4\dot k\dot\lambda k\lambda)\,\dot\vartheta'-x\,(1+x^2)\,\dot\vartheta = -2k\lambda e^{\frac{1}{2}x^2}x\theta \quad \ldots\ldots\ldots\ldots[1],$$

$$(x^4-1-4\dot k\dot\lambda k\lambda)\,\dot\theta'-x\,(1+x^2)\,\dot\theta = \quad 2k\lambda e^{\frac{1}{2}x^2}x\vartheta \quad \ldots\ldots\ldots\ldots[2],$$

$$(x^4-1-4\dot k\dot\lambda k\lambda)\,\theta'+x\,(1-x^2)\,\theta = -2\dot k\dot\lambda e^{-\frac{1}{2}x^2}x\dot\vartheta \quad \ldots\ldots\ldots[3],$$

$$(x^4-1-4\dot k\dot\lambda k\lambda)\,\vartheta'+x\,(1-x^2)\,\vartheta = \quad 2\dot k\dot\lambda e^{-\frac{1}{2}x^2}x\dot\theta \quad \ldots\ldots\ldots[4].$$

63. Put $x^4-1-4\dot k\dot\lambda k\lambda = X$, then $\vartheta\,[1]+\theta\,[2]$ gives

$$x\,(\theta\dot\theta'+\vartheta\dot\vartheta')-x\,(1+x^2)(\theta\dot\theta+\vartheta\dot\vartheta) = x\Theta_1-x\,(1+x^2)\,\Theta_0 = 0\ldots[5],$$

and in like manner $\dot\theta\,[3]+\dot\vartheta\,[4]$ gives

$$x\,(\theta'\dot\theta+\vartheta'\dot\vartheta)+x\,(1-x^2)(\theta\dot\theta+\vartheta\dot\vartheta) = x\Theta_2+x\,(1-x^2)\,\Theta_0 = 0\ldots[6].$$

64. And from $\dot\theta'\,\{1\}+\dot\vartheta'\,\{2\}$, see Art. 59, we get

$$-(1+x^2)(\theta'\dot\theta'+\vartheta'\dot\vartheta')+x\,(\theta\dot\theta'+\vartheta\dot\vartheta') = -(1+x^2)\,\Theta_3+x\Theta_1 = 0\ldots[7],$$

and from $\theta'\,\{3\}+\vartheta'\,\{4\}$ we find

$$-(1-x^2)(\theta'\dot\theta'+\vartheta'\dot\vartheta')-x^2\,(\theta'\dot\theta+\vartheta'\dot\vartheta) = -(1-x^2)\,\Theta_3-x\Theta_2 = 0\ldots[8].$$

65. Now, recurring to Art. 60, and therein replacing m by $\dot k k$, we have

$$\begin{aligned}4\dot k\dot\lambda k\lambda\Theta_3 &= (1-x^4)\,\Theta_3+x\,(1+x^2)\,\Theta_2-x\,(1-x^2)\,\Theta_1-x^2\Theta_0\\ &= x\,(1+x^2)\,\Theta_2-x^2\Theta_0 = -(1-x^4)\,\Theta_3-x^2\Theta_0,\end{aligned}$$

by [7] and [8],

therefore $\quad (x^4-1-4\dot k\dot\lambda k\lambda)\,\Theta_3-x^2\Theta_0 = X\Theta_3-x^2\Theta_0 = 0,$

and $$\Theta_0 = \frac{X}{x^2}\,\Theta_3, \quad \Theta_1 = \left(x+\frac{1}{x}\right)\Theta_3,$$

and $$\Theta_2 = \left(x-\frac{1}{x}\right)\Theta_3.$$

66. Differentiate the first of the equations of Art. 58, and eliminate A', B', C', and D' by means of (a), (b), (c), and (d) of Art. 9. Then we see that, identically,

$$\Theta_0' = \Theta_1+\Theta_2;$$

moreover, by Art. 65, we have

$$\Theta_1+\Theta_3=2x\Theta_3;$$

hence $$\Theta_0'=2x\Theta_3,$$

and (see Art. 65) $$\frac{\Theta_0'}{\Theta_0}=\frac{2x^3}{X}=\frac{2x^3}{x^4-1-4\dot{k}\dot{\lambda}k\lambda};$$

and $$\Theta_0=C\surd X.$$

67. Equating the values of Θ_0, in Arts. 65 and 66,

$$\frac{X}{x^2}\Theta_3=C\surd X,$$

$$\Theta_3^2X=\Theta_3^2(x^4-1-4\dot{k}\dot{\lambda}k\lambda)=C^2x^4;$$

and $$C^2=\Theta_3^2,$$

and $$1+4\dot{k}\dot{\lambda}k\lambda=0=1+4m\dot{\lambda}\lambda.$$

68. Hence the system of equations of Art. 58 may now be written

$$x^2D-2x(B+C)+4A=\Theta_0=Mx^2,$$

$$x^2G-xD-2xE+2B=\Theta_1=M\left(x+\frac{1}{x}\right),$$

$$x^2H-xD-2xF+2C=\Theta_2=M\left(x-\frac{1}{x}\right),$$

$$x^2I-xG-xH+D=\Theta_3=M=\tfrac{1}{2}\lambda=-\tfrac{1}{2}\dot{\lambda};$$

where M and Θ_3 mean the same expression, but M is used to remind us that the Θ's have a double character, one as compounds of $A, B, \ldots I$, and the other as functions of M (an arbitrary constant) and x, and that in each they must be shown to fulfil the requisite conditions.

69. It is now evident that the two different forms of $\dot{U}U+\dot{V}V$ give coincident results; for, on making

$$4m\lambda\dot{\lambda}\ (=4\dot{k}\dot{\lambda}k\lambda)=-1,$$

the last equation of Art. 60 becomes, on replacing the Θ's by their respective values,

$$M\{(2-x^4)+x^4-1-(1-x^4)-x^4\}=0,\quad \text{an identity.}$$

As another verification, write the first equation of Art. 53 in the form

$$U' = \left(\frac{1}{x} + x\right) U + 2k\dot{\lambda}\,\frac{e^{\frac{1}{2}x^2}}{x}\,V;$$

then, differentiating, and eliminating U'' by means of Art. 51, U' also disappears, and we get

$$U = 2k\dot{\lambda}e^{\frac{1}{2}x^2}\left\{x\dot{V}' + (x^2-1)\,\dot{V}\right\};$$

differentiate again, and also, by Art. 51, eliminate $\dot{V}'''$, we get

$$U' = 2k\dot{\lambda}e^{\frac{1}{2}x^2}\left\{(x^2+1)\,\dot{V}' + x^3\dot{V}\right\};$$

and, equating these values of U',

$$\left(\frac{1}{x} + x\right) U + 2k\dot{\lambda}e^{\frac{1}{2}x^2}\left\{-(x^2+1)\,\dot{V}' + \left(\frac{1}{x} - x^3\right)\dot{V}\right\};$$

whence, multiplying into x and dividing out $1+x^2$, we get the former value of U.

70. That the conditions imposed on the Θ's are fulfilled when they are expressed as functions of M and x, it is not difficult to see. But they must be fulfilled when the Θ's are expressed in terms of A, B, ... I, or of some of those symbols. Eliminating accented letters from $(x^2D)' - \{2x\,(B+C)\}' + 4A'$, the identity $\Theta_1 + \Theta_2 = \Theta_0'$ is obtained immediately. But that Θ_3' is a constant, is shown in a different manner. Thus,

$$\Theta_3' = (x^2I)' - \{x\,(G+H)\}' + D',$$

and

$$(x^2I)' = x^2I' + 2xI$$

$$= x^2\left\{-\frac{2}{x}\,I + \left(\frac{1}{x^2} - 2\right)H + \frac{2}{x}\,F + \left(\frac{1}{x^2} + 2\right)G - \frac{2}{x}\,E\right\} + 2xI$$

$$-(xG)' = -xG' - G$$

$$= -x\left\{I - \left(\frac{1}{x} - x\right)G + \left(\frac{1}{x^2} - 2\right)D + \frac{2}{x}\,C\right\} - G$$

$$-(xH)' = -xH' - H$$

$$= -x\left\{I - \left(\frac{1}{x} + x\right)H + \left(\frac{1}{x^2} + 2\right)D - \frac{2}{x}\,B\right\} - H,$$

$$D' = G + H,$$

and therefore

$$\Theta_3' = -2xI+(2+x^2)\,G+(2-x^2)\,H-2xE+2xF-\frac{2}{x}D-2C+2B$$

$$= -\frac{2}{x}\Theta_3+\Theta_1-\Theta_2 = M\left(-\frac{2}{x}+x+\frac{1}{x}+\frac{1}{x}-x\right) = 0,$$

and Θ_3 is constant.

71. That $\Theta_1'+\Theta_2' = \Theta_0'' = 2M$ will now be shown to hold. We have

$$\Theta_2' = (x^2H)'-(xD)'-(2xF)'+(2C)',$$

and

$$(x^2H)' = \{x\,(xH)\}' = x\,(xH)'+xH.$$

But $(xH)'$ is known from Art. 70, and

$$(x^2H)' = x^2\left\{I-\left(\frac{1}{x}+x\right)H+\left(\frac{1}{x^2}+2\right)D-\frac{2}{x}B\right\}+2xH$$

$$-(xD)' = -x\,(G+H)-D,$$

$$-(2xF)' = -2x\left\{H-\left(\frac{1}{x}+x\right)F+\left(\frac{1}{x^2}+2\right)C-\frac{2}{x}A\right\}-2F,$$

$$2C' = 2D+2F,$$

therefore

$$\Theta_2' = x^2I-x\,(2+x^2)\,H-xG+2\,(1+x^2)\,D+2\,(1+x^2)\,F$$

$$-\frac{2}{x}(1+2x^2)\,C-2xB+4A$$

$$= \Theta_3+(1+x^2)\left(-xH+D+2F-\frac{2}{x}\,C\right)+x^2D-2xB-2xC+4A$$

$$= \Theta_3-\left(\frac{1}{x}+x\right)\Theta_2+\Theta_0;$$

also, by calculation,

$$\Theta_1' = \Theta_3+\left(x-\frac{1}{x}\right)\Theta_1-\Theta_0;$$

hence

$$\Theta_0'' = \Theta_1'+\Theta_2' = 2\Theta_3-\left(\frac{1}{x}+x\right)\Theta_2-\left(\frac{1}{x}-x\right)\Theta_1$$

$$= M\left\{2-\left(x^2-\frac{1}{x^2}\right)-\left(\frac{1}{x^2}-x^2\right)\right\} = 2M,$$

as it ought.

72. It follows that the four equations of Art. 68 are not independent, but are equivalent to two independent conditions only. Hence, eliminating M, we find that

$$x^2\Theta_3 - x(\Theta_1+\Theta_2)+\Theta_0 = 0,$$

or $$x^4 I - 2x^3(G+H)+4x^2 D+2x^2(E+F)-4x(B+C)+4A = 0 \ldots (\mathrm{A}).$$

VII. *Conclusion.*

73. The seven equations (a, b, c, d, 1, 2, 3) of Art. 9, and the two (i., ii.) of Arts. 11, 41, 43, together with (A) just arrived at, constitute a system of ten independent equations, homogeneous in eleven quantities and sufficient to determine all that is sought or possible, viz., ratios. In this paper I shall not attempt the determination. But I shall make some remarks intended to facilitate it.

74. Differentiate (4) of Art. 10, eliminating $\dot{u}'''$ and u''', which are particular values of $\dot{y}'''$ and y'''. After considerable reductions, I get

$$\begin{aligned}(B-\dot{p}A)\,G+(C-pA)\,H+(pB-D)\,F+(\dot{p}C-D)\,E \\ = (B-\dot{p}A)\,\dot{u}''u'+(C-pA)\,\dot{u}'u''+(\dot{p}C-D)\,\dot{u}''u+(pB-D)\,\dot{u}u'' \\ -(E+F)\,\dot{u}'u'+(G+pF)\,\dot{u}'u+(H+\dot{p}E)\,\dot{u}u'-(pH+\dot{p}G)\ldots(5).\end{aligned}$$

75. Such relations as (5), if troublesome to attain, are, when attained, easily verified. They are all of the form

$$Æ = \Sigma . J_{r,s}\,\dot{u}^{(r)}u^{(s)},$$

and since, in Art. 9, it is indifferent whether we eliminate $\dot{u}$ and u, or $\dot{v}$ and v, we have

$$Æ = \Sigma . J_{r,s}\,v^{(r)}v^{(s)}.$$

Hence, adding and adopting the notation of Synthemes, we get

$$2Æ = \Sigma . J_{r,s}\,(r,s),$$

which is an identity. The equations (1, 2, 3, 4, 5), although constructed without reference to this identity, all satisfy it.

76. The mixed integral (which, forgetting that it was the same for the dotted as for the undotted system, I calculated independently for each) must, for our present purpose, be expressed in terms of $\dot{y}$ and y. This is most simply done by first eliminating $\dot{Y}''$ (or Y''), and then substituting for $\dot{Y}'$ and $\dot{Y}$ (or Y' and Y). It may be written in the

form

$$x^5\dot{y}''y''+(x^2-2x^4)\,\dot{y}''y'-(x^3+2x^4)\,\dot{y}'y''+2\,(x^3-x)\,\dot{y}''y+2\,(x^3+x)\,\dot{y}y''$$
$$+4x^3\dot{y}'y'+2\,(1-2x^2)\,\dot{y}'y-2\,(1+2x^2)\,\dot{y}y'+4x\dot{y}y=\frac{\dot{c}}{x}=-\frac{\dot{c}}{x};$$

from which differentiation will elicit no additional result, because its sinister multiplied into x and then differentiated vanishes identically, as it ought.

77. This mixed integral also vanishes identically when $\dot{y}=x^3$ and when $y=x^2$. And if we deal with it by the process of Art. 75, replacing $\dot{y}''y''$ and $\dot{y}''y'$, &c., by the Synthemes (2, 2) and (2, 1), &c., and reducing, we get

$$x^3\Theta_3-x^2\Theta_2-x^2\Theta_1+x\Theta_0+\Theta_2-\Theta_1=\frac{2\dot{c}}{x}=-\frac{2\dot{c}}{x},$$

which, by (A) and Art. 68, is

$$-\frac{2M}{x}=\frac{2\dot{c}}{x}=-\frac{2\dot{c}}{x};$$

and

$$\dot{c}=-\dot{c}=M=\Theta_3.$$

I shall write the mixed integral in the forms

$$M=x^5\dot{y}''y''+x^3\,(1-2x^2)\,\dot{y}''y'+\ldots+4x\dot{y}y=\Sigma\,.\,K_{r,s}\,\dot{u}^{(r)}u^{(s)}\ldots,$$

interchanging y and u without further remark.

78. And now arises a process by which any attempt to form directly an algebraical equation in any of the ratios, say $\frac{u'}{u}$, will perhaps be superseded, and which gives a new theorem in the inverse method of differential resolvents.

79. The six equations (1, 2, ... 6) will give five distinct equations of the form

$$M\Sigma\,.\,J_{r,s}\,\dot{u}^{(r)}u^{(s)}=Æ\Sigma\,.\,K_{r,s}\,\dot{u}^{(r)}u^{(s)},$$

which, divided by $\dot{u}u$, will give five non-homogeneous equations in

$$A,\ B,\ \ldots\ I,\ \dot{u}_1,\ \dot{u}_2,\ u_1,\ u_2,$$

where $\dot{u}_1,\ \dot{u}_2,\ u_1,\ u_2=\frac{\dot{u}'}{u},\ \frac{\dot{u}''}{u},\ \frac{u'}{u},\ \frac{u''}{u}$ respectively.

Eliminate $\dot{u}_1$, $\dot{u}_2$, u_1 and u_2, and we get a result of the form

$$f_0(A, B, \ldots I, M, x) = 0.$$

80. Next eliminate I, G, H and D by means of the formulæ in Art. 68. The result will be, say,

$$f_1(A, B, C, E, F, M, x) = 0,$$

which will be changed into, say,

$$f_2(\xi_1, \psi_1, \xi_2, \psi_2, A, M, x) = 0$$

by the assumptions

$$\xi_1 = \frac{1}{x} B - \frac{2}{x^2} A, \quad \psi_1 = x^4 E - x^3 B,$$

$$\xi_2 = \frac{1}{x} C - \frac{2}{x^2} A, \quad \psi_2 = x^4 F - x^3 C.$$

81. But, by differentiating, and substituting for $A', B', \ldots F'$, we get

$$\xi_1' = \frac{1}{x^5}(\psi_1 + Mx^4), \quad \psi_1' = \left(\frac{4}{x} + x\right)\psi_1 - x^5\xi_1 + M(x^3 + x),$$

$$\xi_2' = \frac{1}{x^5}(\psi_2 + Mx^4), \quad \psi_2' = \left(\frac{4}{x} - x\right)\psi_2 + x^5\xi_2 + M(x^3 - x).$$

82. These results enable us to calculate $f_2', f_2'', \ldots$ with comparative ease; but they also lead to a new conclusion. For, by one more differentiation of the last group, we get

$$\xi_1'' + \left(\frac{1}{x} - x\right)\xi_1' + \xi_1 = M\left(\frac{1}{x^4} + \frac{1}{x^2} - 1\right),$$

$$\xi_2'' + \left(\frac{1}{x} + x\right)\xi_2' - \xi_2 = M\left(-\frac{1}{x^4} + \frac{1}{x^2} + 1\right),$$

$$\psi_1'' - \left(\frac{9}{x} + x\right)\psi_1' + \left(\frac{24}{x^2} + 5\right)\psi_1 = -M(4 + 2x^2 + x^4),$$

$$\psi_2'' - \left(\frac{9}{x} - x\right)\psi_2' + \left(\frac{24}{x^2} - 5\right)\psi_2 = M(4 - 2x^2 + x^4),$$

the one system being the correlate of the other; and the four bi-ordinals are, respectively, the Differential Resolvents of four alge-

braical equations of the respective forms

$$\phi_1(\xi_1, M, x) = 0, \quad \phi_2(\psi_1, M, x) = 0, \quad \phi_3(\xi_2, M, x) = 0,$$

and

$$\phi_4(\psi_2, M, x) = 0.$$

83. The connection of the results of this paper with the Riccatian

$$z'' + x^{1/i - 2} z = 0,$$

where i is any integer, is made in § V. The theory of Differential Resolvents only makes its appearance collaterally, otherwise many remarks would arise upon the above forms, which indeed determine those of the roots of the algebraical equations.

On Polygons inscribed in a Quadric and circumscribed about two Confocal Quadrics. *By* R. A. Roberts.

[*Read Feb.* 10*th*, 1887.]

1. In a paper recently published in the *Proceedings of the London Mathematical Society* (Vol. xvi., p. 242), I noticed the possibility of the inscription of a doubly infinite number of polygons in a quadric such that the sides touch two confocal quadrics, provided these three quadrics are connected by a certain pair of relations depending on the number of sides of the polygon. This result was arrived at by making use of Liouville's form of the differential equations of the system of lines touching two confocal quadrics (*Journal de Mathématiques*, t. xii., p. 418), and was seen to depend upon the finding of the $2n^{\text{th}}$ parts of the complete values of the hyperelliptic integrals $L(x)$, $M(x)$.

2. The object of the present paper is to show how to obtain the actual relations connecting the quadrics corresponding to a given number of sides of the polygon. For this purpose I employ, for the sake of symmetry, Professor Cayley's form of the equation of a system of confocal quadrics, namely,

$$\frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and take p, q, r as the roots of this equation in p. Now, it is shown in works on geometry of three dimensions that the tangent cone to the surface $p = p_1$ drawn from the point p, q, r, and referred to the normals to the three latter confocals, is

$$\frac{x^2}{p-p_1} + \frac{y^2}{q-p_1} + \frac{z^2}{r-p_1} = 0.$$

Hence, taking two confocals $p = p_1$, $p = p_2$, and solving between the two equations, we have for a common tangent passing through the vertex of the cone, after having transformed to polar coordinates,

$$\cos\alpha = \sqrt{\left\{\frac{(p-p_1)(p-p_2)}{(p-q)(p-r)}\right\}}, \quad \cos\beta = \sqrt{\left\{\frac{(q-p_1)(q-p_2)}{(q-p)(q-r)}\right\}},$$

$$\cos\gamma = \sqrt{\left\{\frac{(r-p_1)(r-p_2)}{(r-p)(r-q)}\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But if $d\sigma$ is an element of the line, and $d\sigma_1$, $d\sigma_2$, $d\sigma_3$ the elements of the lines of curvature of the confocals, we have

$$\cos\alpha = \frac{d\sigma_1}{d\sigma}, \quad \cos\beta = \frac{d\sigma_2}{d\sigma}, \quad \cos\gamma = \frac{d\sigma_3}{d\sigma},$$

and we know that

$$d\sigma_1 = \tfrac{1}{2}\sqrt{\left\{\frac{(p-q)(p-r)}{(p+a)(p+b)(p+c)}\right\}}\,dp, \; d\sigma_2 = \&c., \; d\sigma_3 = \&c.\ldots\ldots(3).$$

Hence, from (2) and (3), we easily find

$$\frac{dp}{\sqrt{P}} + \frac{dq}{\sqrt{Q}} + \frac{dr}{\sqrt{R}} = 0, \quad \frac{p\,dp}{\sqrt{P}} + \frac{q\,dq}{\sqrt{Q}} + \frac{r\,dr}{\sqrt{R}} = 0,$$

where
$$P = (p-p_1)(p-p_2)(p+a)(p+b)(p+c),$$
$$Q = (q-p_1)(q-p_2)(q+a)(q+b)(q+c),$$
$$R = (r-p_1)(r-p_2)(r+a)(r+b)(r+c).$$

These are Liouville's equations, and by their integration we obtain

$$\left.\begin{aligned} L(p) \pm L(q) \pm L(r) &= c_1 \\ M(p) \pm M(q) \pm M(r) &= c_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where
$$L(p) = \int \frac{dp}{\sqrt{P}}, \quad M(p) = \int \frac{p\,dp}{\sqrt{P}},$$

and c_1, c_2 are constants. Putting now $L(q) \pm L(r) = u$, $M(q) \pm M(r) = v$,

we have, for the two points where the line intersects the quadric $p =$ a constant,

$$u_1 - u_2 = 2L(p), \quad v_1 - v_2 = 2M(p) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence we see that the lines of the system will form a doubly infinite number of polygons of n sides inscribed in the quadric p, if we have

$$2nL(p) = \Omega, \quad 2nM(p) = \Omega' \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where Ω, Ω' are the periodic values of the integrals $L(p)$, $M(p)$, respectively.

3. What we want to do is, then, to express these relations in an algebraic form; and for this purpose I make use of the following particular case of Abel's theorem: Let ϕ, ψ, X be polynomials in x, of the n^{th}, $(n-3)^{\text{th}}$, and sixth degrees, respectively; then, if

$$\phi^2 - \psi^2 X = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

there is

$$\Sigma \int \frac{dx}{\sqrt{X}} = \Omega, \qquad \Sigma \int \frac{x\,dx}{\sqrt{X}} = \Omega' \quad \ldots\ldots\ldots\ldots\ldots (8),$$

where the summation refers to all the roots of $f(x)$, and the origin of the integrals is one of the roots of X. If we now suppose all the roots of $f(x)$ to coincide, we come upon the equations (6) obtained above.

4. Let us consider the case of the triangle, that is, $n=3$; then ϕ is a cubic expression, ψ a constant, and $f(x) = m^2(x-z)^6$, say, and (7) may be written $\{\phi - m(x-z)^3\}\{\phi + m(x-z)^3\} = X$. Hence, if U_1, U_2 are two cubic factors of X, we must have $\phi - m(x-z)^3 = kU_1$, $\phi + m(x-z)^3 = k^{-1}U_2$, from which, eliminating ϕ, we get

$$2mk(x-z)^3 = U_2 - k^2 U_1.$$

We see thus that X breaks up into two cubics, U_1, U_2, whose combinantive invariant Q vanishes; that is, they are such that $lU_1 + mU_2$ can be made a perfect cube, the root of the cube determining the parameter x corresponding to (6), when $n = 3$. Applying this result to (4), we have one of the roots of P infinite, and if we take for the two cubics those whose roots are ∞, p_1, p_2 and $-a, -b, -c$, respectively, we may write

$$(x+a)(x+b)(x+c) + \lambda(x-p_1)(x-p_2) = \mu(x-p)^3 \ldots\ldots\ldots (9).$$

Substituting, then, in this identity, $-a$, $-b$, $-c$ successively for x, and eliminating λ, μ, we obtain

$$\frac{(p+a)^3}{(p_1+a)(p_2+a)} = \frac{(p+b)^3}{(p_1+b)(p_2+b)} = \frac{(p+c)^3}{(p_1+c)(p_2+c)} \quad \ldots \quad (10).$$

From this result it appears that, given the quadric p, the quadrics p_1, p_2 are uniquely determined, their parameters then being connected by the equation

$$(b-c)\sqrt[3]{\{(a+p_1)(a+p_2)\}} + (c-a)\sqrt[3]{\{(b+p_1)(b+p_2)\}} + (a-b)\sqrt[3]{\{(c+p_1)(c+p_2)\}} = 0;$$

but, given one of the inscribed quadrics, there are three corresponding circumscribing quadrics, only one of which is, however, real, as is evident from the equation

$$p = p_1 - \sqrt[3]{\{(p_1+a)(p_1+b)(p_1+c)\}},$$

which may be derived directly from (9) by putting $x = p_1$; for from the coefficients of x^3 we see that $\mu = 1$.

5. Again, taking ∞, $-a$, $-b$, and p_1, p_2, $-c$, as the roots of the two cubics respectively, we have

$$\lambda(x+a)(x+b) + (x+c)(x-p_1)(x-p_2) = \mu(x-p)^3 \quad \ldots \quad (11),$$

from which, by comparing the coefficients of x^3, we get $\mu = 1$; and then, putting $-a$, $-b$ successively for x, we find

$$(a-c)(a+p_1)(a+p_2) = (a+p)^3, \quad (b-c)(b+p_1)(b+p_2) = (b+p)^3 \ldots (12).$$

In this case then, as in the foregoing, being given the circumscribing quadric, the two inscribed are uniquely determined, and, being given one of the inscribed, there is one real circumscribed quadric.

By interchanging the constants a, b, c, in (12), we have two further pairs of conditions, namely,

$$(a-b)(a+p_1)(a+p_2) = (a+p)^3, \quad (c-b)(c+p_1)(c+p_2) = (c+p)^3;$$

$$(b-a)(b+p_1)(b+p_2) = (b+p)^3, \quad (c-a)(c+p_1)(c+p_2) = (c+p)^3.$$

6. Let us consider now the two cubics whose roots are ∞, p_1, $-c$, and p_2, $-a$, $-b$, respectively; then we have

$$\lambda(x-p_1)(x+c) + (x-p_2)(x+a)(x+b) = \mu(x-p)^3 \ldots\ldots \quad (13),$$

whence we get $\mu = 1$; and, by putting p_1 and $-c$ successively for x,

there results

$$(p_1-p_2)(p_1+a)(p_1+b)=(p_1-p)^3,\quad (p_2+c)(a-c)(b-c)=(p+c)^3 \qquad \ldots\ldots\ldots (14).$$

Also, putting $-a$, $-b$ successively for x in (13), and eliminating λ, we obtain

$$(a-c)(a+p_1)(b+p)^3-(b-c)(b+p_1)(a+p)^3=0 \qquad \ldots\ldots (15),$$

which relation must of course be implied in the other two.

Given the surface p, then we see, from (14) and (15), that the inscribed quadrics are uniquely determined; but, when one of the inscribed surfaces is given, there are two real circumscribing surfaces, arising from the fact that the connecting relations are not symmetrical in p_1, p_2.

There are evidently two other pairs of relations to be obtained by interchanging a, b, c in (14) and (15). We have thus, altogether, seven distinct pairs of relations connecting the quadrics in the case of the triangle.

7. We may now proceed to consider the quadrilateral. In this case, ϕ is of the fourth degree, and ψ linear in x; also $f(x)=m^2(x-z)^8$;

hence we have $$\phi^2-m^2(x-z)^8=\psi^2 U_2 U_4 \qquad \ldots\ldots\ldots\ldots (16),$$

where U_2 is a quadratic, and U_4 a quartic factor of X.

Thus, in the same manner as before, we obtain

$$\phi+m(x-z)^4=U_4,\quad \phi-m(x-z)^4=\psi^2 U_2;$$

therefore $$2m(x-z)^4=U_4-\psi^2 U_2 \qquad \ldots\ldots\ldots\ldots (17),$$

from which equation all the required results are obtainable.

Taking p_1, p_2 as the roots of U_2, and ∞, $-a$, $-b$, $-c$ as those of U_4, we have

$$\lambda(x+a)(x+b)(x+c)+(sx+t)^2(x-p_1)(x-p_2)=(x-p)^4 \ldots (18),$$

if we suppose ψ equal to $sx+t$.

From the coefficients of x^4 we have then $s=1$, and, by putting $-a$, $-b$, $-c$ successively for x, we find

$$\frac{(a+p)^2}{\sqrt{\{(a+p_1)(a+p_2)\}}}=t-a,\quad \frac{(b+p)^2}{\sqrt{\{(b+p_1)(b+p_2)\}}}=t-b,$$

$$\frac{(c+p)^2}{\sqrt{\{(c+p_1)(c+p_2)\}}}=t-c \ldots\ldots\ldots\ldots (19),$$

whence, by the elimination of t, we get two relations connecting p_1, p_2, p.

Eliminating p_1, p_2 between these equations, we obtain

$$\frac{4}{t+p} = \frac{1}{t-a} + \frac{1}{t-b} + \frac{1}{t-c},$$

which gives three values of t for a given value of p. Hence, being given the surface p, there are three pairs of inscribed surfaces. Also, given p_1, to find p we have from (18) the discriminant of

$$(t+p)^4 + \lambda (t-a)(t-b)(t-c),$$

with respect to t equal to zero; and then, putting p_1 for x in the same equation, there results

$$\lambda (p_1+a)(p_1+b)(p_1+c) = (p-p_1)^4.$$

We thus obtain an equation of the tenth degree in p_1, after having divided out by $(p-p_1)^6$.

8. Again, let us take $-a$, $-b$, $-c$, p_1 as the roots of U_4, and ∞, p_2 as those of U_2; we have then, from (17),

$$\lambda (x-p_1)(x+a)(x+b)(x+c) - \mu (x+t)^2 (x-p_2) = (x-p)^4 \ldots (20),$$

in which we find $\lambda = 1$, and then readily obtain

$$\frac{(b-c)(a+p)^2}{\sqrt{(a+p_2)}} + \frac{(c-a)(b+p)^2}{\sqrt{(b+p_2)}} + \frac{(a-b)(c+p)^2}{\sqrt{(c+p_2)}} = 0 \quad \ldots (21),$$

$$(p_2-p_1)(a+p_2)(b+p_2)(c+p_2) - (p-p_2)^4 = 0 \ldots\ldots\ldots (22).$$

Hence, being given the surface p, we have from (21) four surfaces such as p_2, and then from (22) there are four corresponding surfaces such as p_1.

9. Furthermore, let $-a$, $-b$, ∞, p_1 be the roots of U_4, and $-c$, p_2 those of U_2; then we have

$$\lambda (x+a)(x+b)(x-p_1) + (x+c)(x-p_2)(sx+t)^2 = (x-p)^4 \ldots (23),$$

whence we obtain $s = 1$, and

$$\left.\begin{aligned} \frac{(a+p)^2}{\sqrt{\{(a-c)(a+p_2)\}}} - \frac{(b+p)^2}{\sqrt{\{(b-c)(b+p_2)\}}} &= a-b \\ (a-c)(b-c)(p_1+c)(p-p_2)^4 - (a+p_2)(b+p_2)(p_1-p_2)(p+c)^4 &= 0 \end{aligned}\right\} (24).$$

From the first of these equations we have four values of p_2 corresponding to a given value of p, and from the second there is then a single value of p_1 for each value of p_2. In this case, there are of

course three pairs of equations such as (24) altogether, which are found by interchanging a, b, c.

10. Again, let us take $-a$, $-b$, p_1, p_2 as the roots of U_4, and c, ∞ as those of U_2; we have then

$$(x+c)(sx+t)^2+\lambda(x+a)(x+b)(x-p_1)(x-p_2)=(x-p)^4 \ldots (25),$$

whence $\lambda = 1$, and

$$(a-c)(b-c)(c+p_1)(c+p_2)=(c+p)^4 \ldots\ldots\ldots (26),$$

$$\left.\begin{aligned}\frac{(p_1+b)(a+p)^2}{\sqrt{(c-a)}}-\frac{(p_1+a)(b+p)^2}{\sqrt{(a-b)}}&=\frac{(a-b)(p-p_1)^2}{\sqrt{(p_1+c)}}\\ \frac{(p_2+b)(a+p)^2}{\sqrt{(c-a)}}-\frac{(p_2+a)(b+p)^2}{\sqrt{(c-b)}}&=\frac{(a-b)(p-p_2)^2}{\sqrt{(p_2+c)}}\end{aligned}\right\} \ldots (27),$$

which are of course only equivalent to two equations. From these equations it appears that, when we are given p, there are three pairs of inscribed surfaces. Also, if we are given one of the inscribed surfaces, there are eight corresponding circumscribed surfaces.

In the preceding case there are three different sets of conditions found by interchanging a, b, c; and we thus have altogether, in the case of the quadrilateral, eight distinct pairs of conditions.

11. We may now proceed to consider the case of the pentagon.

We have then either $(x-z)^5=\psi_1^2 U+\psi_2^2 V \ldots\ldots\ldots (28),$

or $(x-z)^5=\psi^2 U_1+U_5 \ldots\ldots\ldots (29),$

where ψ_1, ψ_2 are the factors of the quadratic ψ, U, V are cubic factors of X, and U_1 is a linear factor and U_5 the remaining factor of the fifth degree.

Let α, β, γ be the roots of U, and α', β', γ' those of V; then, from (28), we easily find

$$(\beta'-\gamma')\sqrt{\left\{\frac{(\alpha'-z)^5}{(\alpha'-\alpha)(\alpha'-\beta)(\alpha'-\gamma)}\right\}}$$

$$+(\gamma'-\alpha')\sqrt{\left\{\frac{(\beta'-z)^5}{(\beta'-\alpha)(\beta'-\beta)(\beta'-\gamma)}\right\}}$$

$$+(\alpha'-\beta)\sqrt{\left\{\frac{(\gamma'-z)^5}{(\gamma'-\alpha)(\gamma'-\beta)(\gamma'-\gamma)}\right\}}=0 \ldots\ldots (30),$$

$$(\beta-\gamma)\sqrt{\left\{\frac{(\alpha-z)^5}{(\alpha-\alpha')(\alpha-\beta')(\alpha-\gamma')}\right\}}$$

$$+(\gamma-\alpha)\sqrt{\left\{\frac{(\beta-z)^5}{(\beta-\alpha')(\beta-\beta')(\beta-\gamma')}\right\}}$$

$$+(\alpha-\beta)\sqrt{\left\{\frac{(\gamma-z)^5}{(\gamma-\alpha')(\gamma-\beta')(\gamma-\gamma')}\right\}}=0\ldots\ldots\ldots\ (31).$$

Applying this result to the case we are considering, that is, putting $z=p$, and taking ∞, p_1, p_2, $-a$, $-b$, $-c$ as the roots of X, we get seven pairs of relations connecting p_1, p_2, p. Again, let α be the root of U_1 and α_1, α_2, α_3, α_4, α_5 the roots of U_5; then, substituting these latter quantities for x successively in (29), we get

$$\Sigma'_5\frac{1}{\chi'(\alpha_i)}\sqrt{\left(\frac{(\alpha_i-z)^5}{\alpha_i-\alpha}\right)}=0,\quad \Sigma'_5\frac{x}{\chi'(\alpha_i)}\sqrt{\left(\frac{(\alpha_i-z)^5}{\alpha_i-\alpha}\right)}=0\ldots\ldots(32),$$

where $\chi(x)\equiv U_5$, and the summation refers to the five roots of U_5. From these conditions we get five pairs of relations connecting p_1, p_2, p. We have thus altogether twelve distinct pairs of relations corresponding to the case of the pentagon.

12. Proceeding to the case of the Hexagon, we find, from (7),

$$(x-z)^6=\psi_1^2U_4+\psi_2^2U_2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33),$$

where ψ_1 is a linear and ψ_2 the corresponding quadratic factor of ψ, and U_4 is a biquadratic and U_2 the corresponding quadratic factor of X. If, then, α_1, α_2, α_3, α_4 are the roots of $U_4\equiv\chi(x)$, say, and β_1, β_2 those of U_2, we easily find

$$\Sigma'_4\frac{(\alpha_i-z)^3}{\chi'(\alpha_i)\sqrt{\{(\alpha_i-\beta_1)(\alpha_i-\beta_2)\}}}=0\ldots\ldots\ldots\ldots\ldots\ldots(34),$$

where the summation refers to all the roots of $\chi(x)$. The other relation is of a more complicated nature. A particular form of it can be found thus:

Putting $\psi_2=\lambda+\mu x+\nu x^2$ and $\psi_1=\rho+\sigma x$,

we can find expressions for λ, μ, ν, ρ, σ by the substitution of the roots of χ successively in (33); and then, by the comparison of any of the coefficients of the powers of x, we can get another relation; as, for instance, from the coefficients of x^6 we have

$$\sigma^2+\nu^2=1.$$

We thus find eight distinct pairs of equations altogether in the case of the hexagon.

13. For polygons of a greater number of sides we can find, in the same way, the necessary conditions; but they are not capable of being expressed in the same simple form by means of radicals as in the cases we have considered. It may be observed, however, that the general conditions can be expressed by means of certain determinants. We may suppose $p = 0$, without loss of generality; and then, since

$$\phi - \psi\sqrt{X} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

is to be satisfied by $2n$ equal values of x, we may differentiate this expression $2n-1$ times, and then put $x = 0$. Differentiating (35) $n+1$ times with regard to x, we eliminate all the constants in ϕ, and there is then

$$\frac{d^{n+1}}{dx^{n+1}}(\psi\sqrt{X}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36).$$

Now let us suppose $\sqrt{X}$ expanded in the form

$$A_0 + A_1 x + A_2 x^2 + \&c.,$$

and let $$\psi = \alpha + \beta x + \gamma x^2 + \ldots + \lambda x^{n-3};$$

then, putting $x = 0$, in (36), and its successive $(n-2)^{\text{th}}$ differential coefficients, we get

$$\begin{aligned} \alpha A_{n-2} + \beta A_{n-1} + \gamma A_n \quad + \ldots &= 0, \\ \alpha A_{n-3} + \beta A_{n-2} + \gamma A_{n-1} + \ldots &= 0, \\ \alpha A_{n-4} + \beta A_{n-3} + \gamma A_{n-2} + \ldots &= 0, \\ \qquad\qquad + \ldots &= 0. \end{aligned}$$

That is, we have $n-1$ linear homogeneous equations connecting the $n-2$ quantities α, β, γ, &c.; hence, by elimination, we obtain $n-1$ determinants, which, being equated to zero, represent the two relations sought.

14. In the same way we can demonstrate the possibility of circumscribing an infinite number of polygons about two confocal quadrics, so that each vertex may move on a confocal quadric. In fact, from (5), we find

$$2\Sigma'_n L(p_i) = \Omega, \quad 2\Sigma'_n M(p_i) = \Omega' \quad \ldots\ldots\ldots\ldots\ldots (37),$$

where n is the number of the vertices of the polygon, and the summa-

tion refers to the n confocal quadrics passing through these points. Let us suppose, now, f to be the quantic whose roots are the parameters of the n circumscribing quadrics; then we have

$$\phi^2-\psi^2X=f^2, \text{ or } \phi^2-f^2=\psi^2X;$$

whence, if P, Q are two factors of ψ^2X of the n^{th} degree in x, we find

$$f=\lambda P+\mu Q,$$

from which the two required conditions can be found. For instance, for the triangle, if

$$f=(x-\varpi_1)(x-\varpi_2)(x-\varpi_3),$$

we have $$(x+a)(x+b)(x+c)+\lambda(x-p_1)(x-p_2)=kf,$$

or equations of the form

$$(x-p_1)(x-p_2)(x+c)+\lambda(x+a)(x+b)=kf,$$

$$(x-p_1)(x+a)(x+b)+\lambda(x-p_2)(x+c)=kf,$$

from which the respective pairs of conditions are easily found; and in the same way, for polygons of a greater number of sides, the equations of condition are found with no greater difficulty than in the case in which all the circumscribing surfaces coincide.

15. It may be observed that one or both of the inscribed quadrics can be supposed to degenerate into focal conics, if the sides of the polygon are not less than four in number. In this case the equations of condition are in general considerably simplified. For instance, suppose we wish to find the relations connecting four confocal surfaces, each of which passes through a vertex of a quadrilateral whose sides intersect the two real focal conics; then, putting

$$p_1+a=0, \quad p_2+b=0$$

for the focal conics, the only relevant equation is

$$(x+a)^2(x+b)^2+(sx+t)^2(x+c)=f(x) \quad \ldots\ldots\ldots\ldots(38),$$

where $$f(x)=(x-p_1)(x-p_2)(x-p_3)(x-p_4).$$

We have then $$f(-c)=(a-c)^2(b-c)^2,$$

and the other condition is found by dividing $(x+a)^2(x+b)^2-f(x)$ by $x+c$ and equating to zero the discriminant of the resulting quadratic.

16. In the case in which the vertices of the polygon lie on the same quadric, if the inscribed quadrics coincide, the problem we have been considering becomes—to inscribe in a quadric U polygons such that the sides touch a quadric V along the curve of contact of the common tangent planes of U and V. The two conditions can then be expressed in terms of the roots of the discriminant of $U+\lambda V$. For instance, for the case of the triangle we have, from (10),

$$\frac{a^3}{(p_1+a)^2}=\frac{b^3}{(p_1+b)^2}=\frac{c^3}{(p_1+c)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(39).$$

But if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of the discriminant of $U+\lambda V$, namely,

$$\Delta\lambda^4+\Theta\lambda^3+\Phi\lambda^2+\Theta'\lambda+\Delta'=0 \ldots\ldots\ldots\ldots\ldots\ldots(40)$$

we have

$$\lambda_1=\lambda_4\left(1+\frac{p}{a}\right), \quad \lambda_2=\lambda_4\left(1+\frac{p}{b}\right), \quad \lambda_3=\lambda_4\left(1+\frac{p}{c}\right),$$

whence, from (39), we get $\lambda_1+\lambda_2+\lambda_3=\lambda_4$, and $\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=0$, and then, from (40), $\Theta^2=4\Delta\Phi$, $\Theta\Theta'=2\Delta\Delta'$. For the quadrilateral we find

$$\lambda_1+\lambda_2-\lambda_3-\lambda_4=0, \quad \lambda_1\lambda_2+\lambda_3\lambda_4=0,$$

from which we get $\quad \Theta^2-4\Delta\Phi=0, \quad \Theta'=0$;

or

$$(\lambda_1-\lambda_2)^2=\lambda_3^2+\lambda_4^2+6\lambda_3\lambda_4,$$

$$(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)=(\lambda_3-\lambda_4)^2.$$

17. In the preceding results I have called the system of quadrics confocal, for the sake of convenience; but of course everything remains true if we suppose the quadrics inscribed in the same developable. If the two quadrics referred to their common self-conjugate tetrahedron

are $\quad x^2+y^2+z^2+u^2=0, \quad \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{u^2}{d}=0,$

the equation of any quadric inscribed in their circumscribed developable is

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}+\frac{u^2}{d+\lambda}=0.$$

Then, if p, q, r are the roots of this equation in λ, we have

$$Mx^2=\frac{(a+p)(a+q)(a+r)}{(a-b)(a-c)(a-d)}, \quad My^2=\&c. \ldots\ldots\ldots\ldots\ldots(41).$$

We may now give here a purely algebraical proof of the two differential equations of the lines touching two quadrics of the system. Let L be a sextic in a variable λ, and ϕ, ψ a cubic and a linear

expression, respectively, in the same variable; then, by a particular case of Abel's theorem, if $\phi^2-\psi^2 L = f(\lambda)$, we have

$$\Sigma_6' \int \frac{d\lambda}{\sqrt{L}} = \Omega, \qquad \Sigma_6' \int \frac{\lambda d\lambda}{\sqrt{L}} = \Omega' \ldots\ldots\ldots\ldots\ldots\ldots (42),$$

where Ω, Ω' have the same meaning as at (6), and the summation refers to the six roots of $f(\lambda)$.

Now let $\quad L \equiv (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda+a)(\lambda+b)(\lambda+c)(\lambda+d),$

and suppose the roots of $f(\lambda)$ to be the parameters p, q, r, p', q', r', as found above, of two points in space; then, if we have (42), the six parameters are the roots of $\phi^2-\psi^2 L = 0$. But, substituting the six quantities p, q, r, p', q', r' successively in $\phi-\psi\sqrt{L} = 0$, and eliminating the constants in ϕ, ψ linearly, we get two equations which may be written

$$\left.\begin{aligned} \Sigma \frac{\sqrt{f(-a)}}{(a+\lambda_1)\chi'(a)} &= \frac{\sqrt{f(\lambda_1)}}{\chi(\lambda_1)} \\ \Sigma \frac{\sqrt{f(-a)}}{(a+\lambda_2)\chi'(-a)} &= \frac{\sqrt{f(\lambda_2)}}{\chi(\lambda_2)} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (43),$$

where the summation refers to the four quantities a, b, c, d, and $\chi(x) = (x+a)(x+b)(x+c)(x+d)$. Transforming now by means of (41), and making use of the relation

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}+\frac{u^2}{d+\lambda} \equiv U = \frac{(\lambda-p)(\lambda-q)(\lambda-r)}{\chi(\lambda)},$$

the equations (43) become

$$\frac{xx'}{a+\lambda_1}+\frac{yy'}{b+\lambda_1}+\frac{zz'}{c+\lambda_1}+\frac{uu'}{d+\lambda_1} = \sqrt{U_1 U_1'},$$

$$\frac{xx'}{a+\lambda_2}+\frac{yy'}{b+\lambda_2}+\frac{zz'}{c+\lambda_2}+\frac{uu'}{d+\lambda_2} = \sqrt{U_2 U_2'}.$$

But these are, it is easy to see, the conditions that the lines joining the points $xyzu$, $x'y'z'u'$ should touch the quadrics U_1, U_2. Hence, from (42), if we suppose one of the points to remain fixed, we see that the differential equations of the lines are

$$\frac{dp}{\sqrt{L(p)}}+\frac{dq}{\sqrt{L(q)}}+\frac{dr}{\sqrt{L(r)}} = 0,$$

$$\frac{p\,dp}{\sqrt{L(p)}}+\frac{q\,dq}{\sqrt{L(q)}}+\frac{r\,dr}{\sqrt{L(r)}} = 0.$$

On the Binomial Equation $x^p-1=0$: *Quinquisection.* *By* H. W. Lloyd Tanner, M.A., Professor of Mathematics in the University College of South Wales.

[*Read Feb.* 10*th*, 1887.]

The binomial equation has recently been discussed by Prof. Cayley in three papers published in the *Proceedings of the London Mathematical Society*; viz.,

Vol. xi. (1879), pp. 4—17 .....(𝔄),

Vol. xii. (1880), pp. 15, 16 .....(𝔅),

Vol. xvi. (1885), pp. 61—63 .....(ℭ).

A further contribution to the question is contained in a paper by Miss Charlotte A. Scott, D.Sc., published in the *American Journal of Mathematics* (1886), Vol. viii., pp. 261—264.....(𝔇).

If X_0, X_1, X_2, X_3, X_4 denote the five periods (when $p=5\lambda+1$), we have

$$X_0^2=(a, b, c, d, e\,\mathfrak{X}\,X_0, X_1, X_2, X_3, X_4),$$

$$X_0X_1=(f, g, h, i, j\,\mathfrak{X}\quad\ldots\quad\ldots\quad),$$

$$X_0X_2=(k, l, m, n, o\,\mathfrak{X}\quad\ldots\quad\ldots\quad),$$

and equations derived from these by cyclical permutations. The coefficients a, b, c, ... o are integers, the properties of which are considered in Prof. Cayley's papers; and in (ℭ) they are expressed as linear functions of 5 integers θ, α, β, f, k, which satisfy one linear equation and two quadric equations. In this place is quoted another reduction, due to Prof. F. S. Carey, of the 15 coefficients to 7 integers connected by 3 linear and 2 quadric equations. In that part of Miss Scott's paper which relates to Quinquisection, proofs are given of some of Prof. Cayley's theorems, and Miss Scott finds the values of all but one of the coefficients of the equation whose roots are the five periods.

In the following discussion the theory is based on five integers q_0, q_1, q_2, q_3, q_4, the "coordinates" of a certain complex factor of p; viz.,

$$q\omega=q_0+q_1\omega+q_2\omega^2+q_3\omega^3+q_4\omega^4,$$

where ω is a root of the equation

$$\omega^4+\omega^3+\omega^2+\omega+1=0.$$

This complex integer is, in fact, the quotient

$$F\omega \,.\, F\omega / F\omega^2,$$

made definite as to its coordinates by the condition

$$1+q_0+q_1+q_2+q_3+q_4=0,$$

$F\omega$ being the resolvent

$$X_0+X_1\omega+X_2\omega^2+X_3\omega^3+X_4\omega^4.$$

The coefficients a, b, c, ... o are expressed linearly in terms of the q; and the relations of these 15 coefficients follow with great ease. The coefficients of the period equation are also determined. It is found convenient here to make use of three numbers, σ_1, σ_2, σ_3, closely related to the q. Tables follow in which are given the values of the q, the σ, and the coefficients of the period equations for values of p up to 1000. I take this opportunity of expressing my thanks to my colleague, Mr. W. F. Pelton, M.A., who has kindly verified the Tables for me.

The periods, X, depend, as far as their arrangement is concerned, upon the primitive root used in their formation. The consideration of the effect of changing this primitive root leads to a set of cyclic permutations different from that mentioned above; one being represented by adding the same number to each subscript, the second by multiplying every subscript by one number; thus leaving X_0 unchanged, while X_1, X_2, X_4, X_3 are cyclically permuted. The double set of substitutions much facilitates the work.

Preliminary.

Let p be a prime number of the form $5\lambda+1$, and x any root of the equation

$$x^{p-1}+x^{p-2}+\ldots+x+1=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The periods X_0, X_1, X_2, X_4, X_3 are the sums

$$\left.\begin{aligned} X_0 &= x \;\; +x^{g^5}+x^{g^{10}}+\ldots+x^{g^{5\lambda-5}} \\ X_1 &= x^{g^1}+x^{g^6}+\ldots\ldots\ldots+x^{g^{5\lambda-4}} \\ X_2 &= x^{g^2}+x^{g^7}+\ldots\ldots\ldots+x^{g^{5\lambda-3}} \\ X_4 &= x^{g^4}+x^{g^9}+\ldots\ldots\ldots+x^{g^{5\lambda-1}} \\ X_3 &= x^{g^3}+x^{g^8}+\ldots\ldots\ldots+x^{g^{5\lambda-2}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where g is any primitive root of p. Each of the lines is characterised by a number, the subscript of X, to which all the exponents of g in the line are congruent, to modulus 5. The last four rows are arranged in an order which will be much used in the sequel, the subscripts forming a geometrical progression 1, 2, 4, 3 ($\equiv 8$).

The functions F are defined by the equations

$$\left.\begin{aligned} F1 &= X_0 + X_1 + X_2 + X_4 + X_3, = -1 \\ F\omega &= X_0 + \omega X_1 + \omega^2 X_2 + \omega^4 X_4 + \omega^3 X_3 \\ F\omega^2 &= X_0 + \omega^2 X_1 + \omega^4 X_2 + \omega^3 X_4 + \omega X_3 \\ F\omega^4 &= X_0 + \omega^4 X_1 + \omega^3 X_2 + \omega X_4 + \omega^2 X_3 \\ F\omega^3 &= X_0 + \omega^3 X_1 + \omega X_2 + \omega^2 X_4 + \omega^4 X_3 \end{aligned}\right\} \quad \ldots\ldots\ldots(3),$$

wherein ω represents any root of

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

It is well known that

$$F\omega \,.\, F\omega^4 = F\omega^2 \,.\, F\omega^3 = p \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Also the quotient

$$F\omega^h \,.\, F\omega^k / F\omega^{h+k}$$

is a complex integer; that is to say, it is of the form

$$A_0 + A_1\omega + A_2\omega^2 + A_4\omega^4 + A_3\omega^3,$$

where the "coordinates" A_0, A_1, A_2, A_4, A_3 are integers. In the case in which $h = k = 1$, we shall call this complex integer

$$q\omega, = q_0 + q_1\omega + q_2\omega^2 + q_4\omega^4 + q_3\omega^3,$$

so that we have the first of the following set of equations,

$$\left.\begin{aligned} F\omega \,.\, F\omega &= q\omega \,.\, F\omega^2 \\ F\omega^2 \,.\, F\omega^2 &= q\omega^2 \,.\, F\omega^4 \\ F\omega^4 \,.\, F\omega^4 &= q\omega^4 \,.\, F\omega^3 \\ F\omega^3 \,.\, F\omega^3 &= q\omega^3 \,.\, F\omega \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

the others being formed by successively doubling the exponents of ω.

These equations determine $q\omega$, but not its coordinates, since, in virtue of (4), $q\omega$ is not changed when any, the same, constant is added to each of the coordinates. To fix these, we assume that the

equations (6) hold good when ω is replaced by $\omega^5, = 1$. This gives

$$q1 = -1 \text{ or } 1+q_0+q_1+q_2+q_4+q_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From (5), (6), we find

$$\left.\begin{aligned} F\omega \,.\, F\omega^3 &= q\omega \,.\, F\omega^4 \\ F\omega^2 \,.\, F\omega \;&= q\omega^2 \,.\, F\omega^3 \\ F\omega^4 \,.\, F\omega^2 &= q\omega^4 \,.\, F\omega \\ F\omega^3 \,.\, F\omega^4 &= q\omega^3 \,.\, F\omega^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The equations (5), (6), (8) enable us to express any rational integral function of the F as a linear function of the F. They give, also, the equations

$$\left.\begin{aligned} (F\omega)^5 &= p \,.\, q\omega \,.\, q\omega \,.\, q\omega^2 \\ p \,.\, F\omega^2 &= \qquad\qquad q\omega^4 \,.\, (F\omega)^2 \\ p^3 \,.\, F\omega^4 &= q\omega^4 \,.\, q\omega^4 \,.\, q\omega^3 \,.\, (F\omega)^4 \\ p^2 \,.\, F\omega^3 &= \qquad q\omega^4 \,.\, q\omega^3 \,.\, (F\omega)^3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

showing that $F\omega^2$, $F\omega^4$, $F\omega^3$ (and therefore also the periods) are rational functions of $F\omega$, which is determined as the fifth root of the complex integer, $p\, q\omega\, q\omega\, q\omega^2$.

From the equations (5), (6), we find

$$p = q\omega \,.\, q\omega^4 = q\omega^2 \,.\, q\omega^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

showing that each $q\omega$ is one of two conjugate factors of p. Writing the products at length,

$$\begin{aligned} p &= q_0^2+q_1^2+q_2^2+q_4^2+q_3^2 \\ &\qquad +(q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0)(\omega+\omega^4) \\ &\qquad +(q_0q_2+q_2q_4+q_4q_1+q_1q_3+q_3q_0)(\omega^2+\omega^3) \\ &= q_0^2+q_1^2+q_2^2+q_4^2+q_3^2 \\ &\qquad +(q_0q_2+q_2q_4+q_4q_1+q_1q_3+q_3q_0)(\omega+\omega^4) \\ &\qquad +(q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0)(\omega^2+\omega^3). \end{aligned}$$

Hence, since the equation (4) for ω is irreducible,

$$\left.\begin{aligned} q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0 &= -\lambda, = -\tfrac{1}{5}(p-1) \\ q_0q_2+q_2q_4+q_4q_1+q_1q_3+q_3q_0 &= -\lambda \\ q_0^2+q_1^2+q_2^2+q_4^2+q_3^2 &= 4\lambda+1 \end{aligned}\right\} \quad \ldots\ldots\ldots (11),$$

of which the first pair are found to be equivalent to Prof. Cayley's quadric relations (℄, p. 62).

The following method of calculating the coordinates is a slight modification of one given in Bachmann's *Lehre von der Kreistheilung* (pp. 93–95); or is easily verified from the discussion in Serret's *Cours d'Algèbre supérieure*, § 552. We set down the values of the expression ind. μ + ind. $(\mu+1)$ for all values of μ from 1 to $\frac{1}{2}(p-1)$ inclusive, or, what is sufficient for our purpose, their residues to modulus 5. Twice the number of times the residue i appears is called A_i; but the last term in the series, viz., ind. $\frac{1}{2}(p-1)$ + ind. $\frac{1}{2}(p+1)$, which is $\equiv 2$ ind. $\frac{1}{2}(p-1)$, is only counted once in forming the A to which it belongs. The expression

$$A_0+A_1\omega+A_2\omega^2+A_4\omega^4+A_3\omega^3$$

is a complex factor of p, having all the properties of $q\omega$, except that the sum of the coordinates is $p-2$. If, then, from each A we subtract λ, $=\frac{1}{5}(p-1)$, we shall obtain the coordinates of $q\omega$. Since λ is even, it follows that one, and only one, of the coordinates of q is odd.

As an example, take the case $p = 11$:

$$\mu = 1, 2, 3, 4, 5, 6,$$

$$\text{ind}_2\,\mu \equiv 0, 1, 3, 2, 4, 4 \pmod 5,$$

$$\text{ind}_2\,\mu + \text{ind}_2\,(\mu+1) \equiv 1, 4, 0, 1, \mid 3 \text{ (,,)},$$

therefore $A_0 = 2,\ A_1 = 4,\ A_2 = 0,\ A_4 = 2,\ A_3 = 1,$

$$q_0 = 0,\ q_1 = 2,\ q_2 = -2,\ q_4 = 0,\ q_3 = -1,$$

$$q\omega = 2\omega - 2\omega^2 - \omega^3.$$

Change of Arbitrary Elements.

In the formation of the periods and the F, three arbitrary elements enter: g, the primitive root of p; ω, the root of (4); and x, the root of (1). We propose to consider the effect of replacing these arbitrary quantities by others.

Suppose that the periods and the functions F have been determined for a particular value of g and a particular value of ω. Let us now change the primitive root, the other elements being the same as before, and denote the new periods and F by accents. The new primitive root is $\equiv g^i \pmod p$, where i is prime to $p-1$, and therefore prime to 5. Hence it follows that the periods are interchanged as

wholes, there is no transference of isolated terms from period to period. In fact, the period

$$X_{ik}, = x \mid g^{ik}+\&c., = x \mid (g^i)^k+\&c., = X'_k.$$

To put this otherwise, we write $i \equiv 2^\iota$, $k \equiv 2^\kappa \pmod 5$, which is permissible, as 2 is a primitive root of 5; and, to avoid complicated subscripts, we write $X_{(\kappa)}$ for X_k when we wish to put in evidence the index κ. With this notation, the fact that X'_k is X_{ik} is expressed by saying that $X'_{(\iota)}$ is $X_{(\iota+\kappa)}$, showing that the new periods consist of the X_0 unchanged and the four periods $X_1 X_2 X_4 X_3$ in the same cyclic order; subjected, in fact, to the substitution

$$(X_1 X_2 X_4 X_3)^\iota.$$

This being so, we have

$$\begin{aligned} F'\omega, &= X'_0+\omega X'_{(0)}+\omega^2 X'_{(1)}+\omega^4 X'_{(2)}+\omega^3 X'_{(3)} \\ &= X_0+\omega X_{(\iota)}+\omega^2 X_{(\iota+1)}+\omega^4 X_{(\iota+2)}+\omega^3 X_{(\iota+3)} \\ &= F\omega^{2^{-\iota}}; \end{aligned}$$

that is, the new F' are $F\omega$, $F\omega^2$, $F\omega^4$, $F\omega^3$ acted on by the substitution

$$(F\omega, F\omega^2, F\omega^4, F\omega^3)^{-\iota}.$$

If we now change the root ω, there is no effect upon the X, and this new change may be such as to undo the effect on the F of the change of primitive root. In fact, taking the new ω' to be ω^{2^ι}, we have

$$F'\omega' = F(\omega')^{2^{-\iota}} = F(\omega^{2^\iota})^{2^{-\iota}} = F\omega.$$

The $q\omega$, being derived from the F alone, will also be unaltered by the double change.

It still remains to consider the effect of a change of the primitive root upon the several coordinates q_0, q_1, q_2, q_4, q_3. The change in $q\omega$ is the same as that in $F\omega$; viz., we have

$$q'\omega = q\omega^{2^{-\iota}}.$$

Hence $$q'_0, q'_1, q'_2, q'_4, q'_3 = q_0, q_{(\iota)}, q_{(\iota+1)}, q_{(\iota+2)}, q_{(\iota+3)}.$$

Consider, for instance, the coefficients of ω in the first equation. On the left it is q'_1; on the right it is q_x, where x is such that $(\omega^{2^{-\iota}})^x = \omega$, so that $x = 2^\iota$. Hence the coordinates are changed in the same manner and to the same extent as the periods. This also follows from the method by which q_0, &c. are calculated.

These results may be expressed in the following form:

Any relation between the periods, the coordinates, the numbers $q\omega$, the functions $F\omega$, and the ω, remains true if the subscripts of all the periods and of the coordinates and the exponents of all the free ω be multiplied by i, leaving the $q\omega$, $F\omega$ unchanged. Here $i = 0, 1, 2, 3, 4$. ... (12).

If we replace the x of equation (1) by another root x^{g^i}, we get a cyclic change in the five periods X_0, X_1, X_2, X_3, X_4 in the arithmetic order; namely,

$$X'_h = X_{h+i},$$

and, as a consequence,

$$\begin{aligned} F'\omega &= X'_0 + \omega X'_1 + \ldots + \omega^{-i} X'_{-i} + \omega^{-i+1} X'_{-i+1} + \ldots \\ &= X_i + \omega X_{i+1} + \ldots + \omega^{-i} X'_0 + \omega^{-i+1} X'_1 + \ldots \\ &= \omega^{-i} F\omega. \end{aligned}$$

Generally, $$F'\omega^h = \omega^{-hi} F\omega,$$

and $q\omega^h$ is unchanged.

The change in each coordinate is not simple, depending upon the $F\omega$ it multiplies as well as upon i.

Thus, *any relation between the periods, the numbers $q\omega$, the functions $F\omega$, and ω, remains true if the subscript of each period be increased by i and every $F\omega^h$ be multiplied by ω^{hi}, the $q\omega$ and the free ω being unaltered.* ... (13).

From this we get some useful results. Thus, if Ξ be any function of the periods, and

$$\Xi = A_0 + A_1 F\omega + A_2 F\omega^2 + A_3 F\omega^3 + A_4 F\omega^4,$$

then $$\Sigma\Xi = 5A_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

if A_0 do not involve ω, $q\omega$. In this case $\frac{1}{5}\Sigma\Xi$ is the absolute term in Ξ. Here $\Sigma\Xi$ is written for the sum of five terms formed from Ξ by adding to the subscripts.

Again, since, from (3), (4),

$$5X_0 = -1 + F\omega + F\omega^2 + F\omega^3 + F\omega^4,$$

it follows that $$\Sigma X_0\Xi = -A_0 + p\,(A_1 + A_2 + A_3 + A_4) \ldots\ldots\ldots\ldots (15),$$

for the expression on the right is the absolute term in $5X_0\Xi$.

We may add for reference, though not related to what has preceded some other results of summation. If a be the sum of the coordinates of a complex function,

$$A\omega, = A_0+A_1\omega+A_2\omega^2+A_4\omega^4+A_3\omega^3,$$

then

$$\Sigma A\omega = 5A_0-a,$$

where the Σ means the sum of the *four* terms formed by successively doubling the exponent of ω. Hence

$$\Sigma q\omega = 5q_0+1, = 5\sigma_1+1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(16),$$

where σ_1 is written for q_0, in order to mark the relation of (16) to (17), (18), *post.*

$$\Sigma\omega^{-h}q\omega = 5q_h+1.$$

$$\Sigma q\omega\, q\omega^2 = 5\sigma_2-1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(17),$$

where σ_2 is the absolute part of $q\omega \,.\, q\omega^2$; or

$$\sigma_2 = q_0^2+q_1q_2+q_2q_4+q_4q_3+q_3q_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(17^*).$$

$$\Sigma q\omega\, q\omega\, q\omega^2 = 5\sigma_3+1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(18),$$

where σ_3 is the absolute part of $q\omega\, q\omega\, q\omega^2$; or

$$\sigma_3 = 3q_0^3+2q_0^2-4\lambda q_0+(q_1+q_4)\,q_1q_4+(q_2+q_3)\,q_2q_3$$
$$+2\,(q_1^2q_2+q_2^2q_4+q_4^2q_3+q_3^2q_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(19).$$

Cayley's Coefficients.

Solving the equations (3), we obtain

$$\left.\begin{aligned} 5X_0 &= (1,\ 1,\ 1,\ 1,\ 1)-1,\ F\omega,\ F\omega^2,\ F\omega^4,\ F\omega^3) \\ 5X_1 &= (1,\ \omega^4,\ \omega^3,\ \omega,\ \omega^2) \qquad ,, \\ 5X_2 &= (1,\ \omega^3,\ \omega,\ \omega^2,\ \omega^4) \qquad ,, \\ 5X_4 &= (1,\ \omega,\ \omega^2,\ \omega^4,\ \omega^3) \qquad ,, \\ 5X_3 &= (1,\ \omega^2,\ \omega^4,\ \omega^3,\ \omega\ ) \qquad ,, \end{aligned}\right\} \ldots\ldots(20).$$

Hence $25X_0^2 = (-1+F\omega+F\omega^2+F\omega^4+F\omega^3)^2$

$$= 1+4p+(-2+2q\omega^4+q\omega^3)\,F\omega+\ldots$$
$$= 1+4p+\{-2+2q_0+q_0+(2q_4+q_2)\,\omega+(2q_3+q_4)\,\omega^2$$
$$+(2q_1+q_3)\,\omega^4+(2q_2+q_1)\,\omega^3\}\,F\omega+\ldots.$$

On the other hand, using Prof. Cayley's notation (p. 214),

$$\begin{aligned} 25X_0^2 = \quad & 5a \,.\, 5X_0 + 5b \,.\, 5X_1 + 5c \,.\, 5X_2 + 5e \,.\, 5X_4 + 5d \,.\, 5X_3 \\ = & -5\,(a+b+c+e+d) \\ & + 5\,(a+b\omega^4+c\omega^3+e\omega+d\omega^2)\,F\omega + \&\text{c.} \end{aligned}$$

Comparing the absolute terms

$$a+b+c+d+e = -\tfrac{1}{5}\,(4p+1) = -4\lambda - 1 \quad \ldots\ldots\ldots\ldots (21),$$

and expressing that the coefficients of $F\omega$ are equal, we get

$$\begin{aligned} 5a &= -2 + 2q_0 + q_0 + C, \\ 5b &= \quad\; 2q_4 + q_2 + C, \\ 5c &= \quad\; 2q_3 + q_4 + C, \\ 5e &= \quad\; 2q_1 + q_3 + C, \\ 5d &= \quad\; 2q_3 + q_4 + C. \end{aligned}$$

To determine C, we add these equations, and make use of (7) and (21).

Thus, $$C = -4\lambda.$$

It is convenient to eliminate q_0 by means of (7). The equation for a then becomes

$$5\,(a+1) = -4\lambda - 3q_1 - 3q_2 - 3q_4 - 3q_3.$$

Again,

$$\begin{aligned} 5X_0 \,.\, 5X_1 = & (-1 + F\omega + F\omega^2 + F\omega^4 + F\omega^3) \\ & \qquad (-1 + \omega^4 F\omega + \omega^3 F\omega^2 + \omega F\omega^4 + \omega^2 F\omega^3) \\ = & \; 1 + p\,(\omega^2 + \omega^4 + \omega^3 + \omega) \\ & + \{-1 - \omega^4 + (\omega^3 + \omega^2)\, q\omega^4 + \omega^2 q\omega^3\}\, F\omega + \&\text{c.} \\ = & \; 1 - p \\ & + \{-1 + q_3 + 2q_1 + (q_0 + q_2 + q_3)\,\omega + (q_0 + q_1 + q_4)\,\omega^2 \\ & \qquad + (-1 + q_3 + 2q_4)\,\omega^4 + (q_0 + q_2 + q_3)\,\omega^3\}\, F\omega + \&\text{c.} \end{aligned}$$

But $$\begin{aligned} 5X_0 \,.\, 5X_1 = & \; 5f \,.\, 5X_0 + 5g \,.\, 5X_1 + 5h \,.\, 5X_2 + 5j \,.\, 5X_4 + 5i \,.\, 5X_3 \\ = & -5\,(f+g+h+i+j) \\ & + 5\,(f+g\omega^4+h\omega^3+j\omega+i\omega^2)\,F\omega + \ldots\,. \end{aligned}$$

Hence, as in the preceding case,

$$\begin{aligned} 5f &= \lambda+2q_1+q_3, \\ 5g &= \lambda+2q_4+q_2, \\ 5h &= \lambda- q_1-q_4, \\ 5j &= \lambda- q_1-q_4, \\ 5i &= \lambda- q_2-q_3. \end{aligned}$$

The expression X_0X_2 is formed from X_0X_1 by doubling the subscripts. The coordinates must be similarly changed. The equation

$$X_0X_1 = fX_0+gX_1+hX_2+jX_4+iX_3$$

becomes $$X_0X_2 = kX_0+mX_2+oX_4+nX_3+lX_1;$$

and the equations for f, g, &c., just written, become

$$\begin{aligned} 5k &= \lambda+2q_2+q_1, \\ 5m &= \lambda+2q_3+q_4, \\ 5o &= \lambda- q_2-q_3, \\ 5n &= \lambda- q_2-q_3, \\ 5l &= \lambda- q_1-q_4. \end{aligned}$$

These results are combined in the following Table. The left side contains the notation used by Prof. Cayley (*C*, p. 62); the right contains the notation quoted in the same place from a dissertation by Prof. F. S. Carey. Throughout, $\lambda = \frac{1}{5}(p-1)$.

Cayley's notation.	λ	q_1	q_2	q_4	q_3	Carey's notation.
$5(a+\lambda+1) =$	1	−3,	−3,	−3,	−3	$= 5(\alpha+1)$
$5\theta =$	3	1,	1,	1,	1	
$5(c+\lambda) = 5k =$	1	1,	2,	.	.	$= 5\gamma$
$5(e+\lambda) = 5g =$	1	.	1,	2,	.	$= 5\epsilon$
$5(d+\lambda) = 5m =$	1	.	.	1,	2	$= 5\delta$
$5(b+\lambda) = 5f =$	1	2,	.	.	1	$= 5\beta$
$5a = 5h = 5j = 5l =$	1	−1,	.	−1,	.	$= 5\rho$
$5\beta = 5i = 5n = 5o =$	1	.	−1,	.	−1	$= 5\sigma$

where the first line, for example, is to be read

$$5\,(a+\lambda+1) = \lambda - 3q_1 - 3q_2 - 3q_4 - 3q_3 = 5\,(a+1).$$

From inspection of this Table, it is clear that θ, a are unaffected by any change of the primitive root; that the coefficients k, g, m, f are cyclically permuted by such change; and that (Cayley's) α, β are interchanged when the index of one primitive root with respect to other $\equiv 2$ or 3 (mod. 5), all this being an immediate consequence of the cyclic changes in the coordinates.

By substituting for α, β, θ, f, k, in Prof. Cayley's quadric equations, their values as given in the above Table, I find that they reduce to

$$(q_1+q_4)^2 + q_1 + q_4 + q_1q_3 - q_2q_3 + q_2q_4 = -\lambda,$$

$$(q_2+q_3)^2 + q_2 + q_3 + q_2q_1 - q_4q_1 + q_4q_3 = -\lambda,$$

and these are equivalent to the first two of (11), as appears on eliminating q_0 therefrom.

The Period Equation.

We proceed to determine the coefficients P, of the equation

$$\eta^5 + \eta^4 + P_2\eta^3 + P_3\eta^2 + P_4\eta + P_5 = 0,$$

whose roots are the periods. P_2, P_3, P_4 have been calculated by Miss Scott (Ⓑ, pp. 261, 262), and the forms given below are equivalent to those obtained by Miss Scott.

Two methods will be used. The first is to form the coefficients P directly from the expressions (20) for the roots; the second consists in forming an equation whose roots are $5X+1$, whence the period equation is immediately derived.

Since $$5X_1 = -1 + \omega^4 F\omega + \omega^3 F\omega^2 + \omega F\omega^4 + \omega^2 F\omega^3,$$

we have, by (15),

$$5\Sigma X_0X_1 = 1-p, \quad = 5\Sigma X_0X_2, \text{ by (12)},$$

therefore $$5P_2 = 5\Sigma\,(X_0X_1 + X_0X_2), \quad = -2\,(p-1),$$

therefore $$P_2 = -2\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (P_2).$$

We have already found

$$25X_0X_1 = 1-p + \left\{-1-\omega^4 + (\omega^3+\omega)\,q\omega^4 + \omega^2 q\omega^3\right\} F\omega + \&c.,$$

therefore, by (13),

$$25X_2X_3 = 1-p+\left\{-\omega^3-\omega^2+(\omega+\omega^4)\,q\omega^4+q\omega^3\right\}F\omega+\&c....(2,3),$$

therefore, by (15),

$$\begin{aligned}25\Sigma X_0X_2X_3 &= p-1+p\left\{-2\Sigma\omega+\Sigma(\omega^4+\omega+1)\,q\omega\right\}\\ &= p-1+p\left\{2+5(q_1+q_4+q_0)+3\right\}\\ &= p-1-5p(q_2+q_3),\end{aligned}$$

therefore $$25\Sigma X_0X_4X_1 = p-1-5p(q_4+q_1),$$

therefore $$\begin{aligned}25P_3, &= -25\Sigma(X_0X_2X_3+X_0X_4X_1)\\ &= -2(p-1)-5p(1+\sigma_1),\end{aligned}$$

therefore $$P_3 = -\tfrac{1}{5}\left\{2\lambda+p(1+\sigma_1)\right\}.....(P_3),$$

where σ_1 is written for q_0, as in (16).

From the expression for $25X_2X_3$, we get, by doubling the subscripts, by (12),

$$25X_4X_1 = 1-p+\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^4+q\omega^3\right\}F\omega+\&c.$$

The product $25X_2X_3\,.\,25X_1X_4$ gives

$$625X_1X_2X_3X_4 = A_0+A_1F\omega+A_2F\omega^2+A_4F\omega^4+A_3F\omega^3,$$

where $$\begin{aligned}A_0 &= (1-p)^2+p\Sigma\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega^4+q\omega^3\right\}\\ &\qquad\times\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega+q\omega^2\right\}\\ &= (1-p)^2+p(-4-2\Sigma q\omega-\Sigma q\omega q\omega^2)\\ &= (1-p)^2+p\left\{-5-10\sigma_1-5\sigma_2\right\}.\end{aligned}$$

Hence, by (14), $$\begin{aligned}125P_4, &= 125\Sigma X_1X_2X_3X_4,\\ &= (p-1)^2-5p(1+2\sigma_1+\sigma_2),\end{aligned}$$

therefore $$P_4 = \tfrac{1}{5}\left\{\lambda^2-\tfrac{1}{5}p(1+2\sigma_1+\sigma_2)\right\}.....(P_4).$$

Also, we have, from the expression for $625X_1X_2X_3X_4$,

$$\begin{aligned}3125P_5 &= -3125X_0X_1X_2X_3X_4\\ &= A_0-p(A_1+A_2+A_4+A_3) = A_0-p\Sigma A_1,\end{aligned}$$

and here A_0 is known.

A_1 consists of three parts. The first arises from the product of the absolute terms of $25X_2X_3$, $25X_4X_1$ into the terms containing $F\omega$, and

is $$= (1-p)\left\{-\omega^3-\omega^2+(\omega+\omega^4)\,q\omega^4+q\omega^3\right.$$
$$\left.-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^4+q\omega^3\right\}$$
$$= (1-p)\,(1-q\omega^4+2q\omega^3).$$

This contributes to ΣA_1 the portion

$$(1-p)\,(4+\Sigma q\omega).$$

The second part of A_1 is the coefficient of

$$F\omega^2 \,.\, F\omega^3, = q\omega^4 \,.\, F\omega,$$

in the product of $25X_2X_3$ and $25X_1X_4$. This gives to ΣA_1 the part

$$\Sigma\left[\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^3+q\omega\right\}\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega+q\omega^2\right\}\right.$$
$$\left.+\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega^3+q\omega\right\}\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega+q\omega^2\right\}\right]q\omega^4$$
$$= 12p+3\,(p+1)\,\Sigma q\omega+3\Sigma q\omega q\omega^2.$$

The third part of A_1 comes from the term involving

$$(F\omega^3)^2, = q\omega^3 \,.\, F\omega,$$

in the product of $25X_2X_3$, $25X_1X_4$. The corresponding part of ΣA_1 is

$$\Sigma\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^2+q\omega^4\right\}\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega^2+q\omega^4\right\}q\omega^3$$
$$= -12p-(2p+1)\,\Sigma q\omega+\Sigma q\omega q\omega^2+\Sigma q\omega q\omega q\omega^2.$$

Hence $$\Sigma A_1 = 4-4p+3\Sigma q\omega+4\Sigma q\omega q\omega^2+\Sigma q\omega q\omega q\omega^2,$$

therefore $$3125P_5, = A_0-p\Sigma A_1$$
$$= 1-10p+5p^2-5p\,\Sigma q\omega-5p\,\Sigma q\omega q\omega^2-p\,\Sigma q\omega q\omega q\omega^2$$
$$= 1-6p+5p^2-5p\,(1+5\sigma_1+5\sigma_2+\sigma_3),$$

therefore $$P_5 = \tfrac{1}{25}\left[\lambda^2-\tfrac{1}{5}\left\{p\,(\sigma_1+\sigma_2)+\tfrac{1}{5}\,(p\sigma_3+\lambda+1)\right\}\right] \ldots\ldots\ldots(P_5).$$

[Miss Scott's formulæ are

$$P_2 = -2\lambda,$$
$$P_3 = \quad 2\lambda^2-p\,(\alpha+\beta),$$
$$P_4 = -\lambda^3+p\left\{\theta\,(\alpha+\beta)-\alpha\beta\right\},$$

where θ, α, β have the values given in the Table on p. 223. Remembering the values of σ_1, σ_2 in terms of the q, viz.,

$$\sigma_1 = q_0, \quad \sigma_2 = q_0^2 + (q_1+q_4)\,(q_2+q_3),$$

there is no difficulty in verifying that these expressions for P are equivalent to those just found.]

Second Form of Period Equation.

The first of the equations (20) may be written

$$Y_0 = 5X_0+1 = F\omega+F\omega^2+F\omega^4+F\omega^3.$$

By raising this to the 2nd, 3rd, 4th, 5th powers, we get expressions for Y_0^2, Y_0^3, Y_0^4, Y_0^5. In order to form the sums ΣY^i, by reason of (14), we only require the part independent of F in the expansion of the right-hand member.

Now, the absolute term in $(F\omega+F\omega^2+\ldots)^i$ arises from the terms involving

$$F\omega^h . F\omega^k \ldots,$$

where $$h+k+ \equiv 0 \pmod 5,$$

and every such product of F is replaced, by means of (5), (6), (8), by a product of some of the quantities p, $q\omega$, $q\omega^2$, $q\omega^4$, $q\omega^3$. In this way, and using (14), we obtain the equations

$$\Sigma Y = 0,$$

$$\Sigma Y^2 = 20p,$$

$$\Sigma Y^3 = 15p\,\Sigma q\omega,$$

$$\Sigma Y^4 = 180p^2+20p\,\Sigma q\omega q\omega^2,$$

$$\Sigma Y^5 = 250p^2\Sigma q\omega+5p\,\Sigma q\omega q\omega q\omega^2.$$

From these power-sums it is easy to find the coefficients of the equation whose roots are the Y, viz.,

$$P_1' = 0,$$

$$P_2' = -10p,$$

$$P_3' = -5p\Sigma q\omega,$$

$$P_4' = 5p^2-5p\Sigma q\omega q\omega^2,$$

$$P_5' = -p\Sigma q\omega q\omega q\omega^2.$$

Hence the period equation may be written

$$(5\eta+1)^5-10p\,(5\eta+1)^3-5p\,\Sigma q\omega\,(5\eta+1)^2$$
$$+5p\,(p-\Sigma q\omega q\omega^2)(5\eta+1)-p\,\Sigma q\omega q\omega q\omega^2 = 0.$$

If herein we replace $\Sigma q\omega$, $\Sigma q\omega q\omega^2$, $\Sigma q\omega q\omega q\omega^2$ by their equivalents

$5\sigma_1+1$, $5\sigma_2-1$, $5\sigma_3+1$, and expand the binomials, we again obtain the values of P_2, P_3, P_4, P_5 given above.

Inverse Problem.

It is a somewhat interesting problem to find q given the period equation; for the process ought to indicate the order as well as the values of the coordinates.

From the values of P_2, P_3, P_4, P_5, we at once obtain p, σ_1, σ_2, σ_3 in succession, and without ambiguity.

Next we find the two sums q_1+q_4, q_2+q_3 as the two roots of a quadratic equation

$$Q^2+(1+\sigma_1)\,Q+\sigma_2=\sigma_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

To complete the determination we have only the equation for σ_3, and two of the quadric relations (11). By using the latter system, we get equations such as

$$\left.\begin{aligned} 8\lambda+2\sigma_2-\ 5\sigma_1^2-2\sigma_1+1 &= (Q_1-2q_1)^2+(Q_2-2q_2)^2 \\ -8\lambda-6\sigma_2+11\sigma_1^2+6\sigma_1+1 &= (Q_1+2q_1)^2+(Q_2+2q_2)^2 \end{aligned}\right\} \ldots\ldots(23),$$

where Q_1, $= q_1+q_4$, and Q_2, $= q_2+q_3$, are the roots of the equation (22). If, then, the numbers on the right be expressed as the sum of two squares in every possible way, we get a number of solutions, from amongst which the proper one is selected by using the σ_3 equation. It goes without saying, that the equations such as (23) indicate relations between the σ and λ; for instance, σ_2 cannot be $\equiv 3$ (mod. 4).

One of the properties of the coordinates deserves notice, as it suggests a simple way of arranging the table. The four coordinates q_1, q_2, q_4, q_3, so far as they appear in the annexed table, are divisible into two distinct groups. In the first and larger of these, the five numbers $\frac{1}{2}\lambda$, $= \frac{1}{10}(p-1)$, q_1, q_2, q_4, q_3 are all of different classes with respect to 5. If we arrange them so that $q_1 \equiv \frac{1}{2}\lambda+1$, then each coordinate $q_i \equiv \frac{1}{2}\lambda+i$ and $q_4 \equiv q_0$. This order is adopted in the table for those values of p to which the process is applicable.

The second group of coordinates comprises those belonging to $p=211$, 281, 421, 461, 521, 691, 881, 991 (marked with an asterisk in the table). In this group all the numbers $\frac{1}{2}\lambda$, q_1, q_2, q_4, q_3 have the same remainder when divided by 5. The coordinate q_0 is excepted, for $q_0 \equiv \frac{1}{2}\lambda-1$ always (for otherwise P_3 would not be integral). In this case I have not found a satisfactory mode of selecting q_1.

TABLE OF q, σ.

p	q_1	q_2	q_4	q_3	$q_0 = \sigma_1$	σ_2	σ_3
11	2	− 2	0	− 1	0	− 6	− 18
31	− 1	0	2	− 4	2	0	− 82
41	0	− 4	3	2	− 2	− 2	196
61	2	3	0	− 6	0	− 6	222
71	− 2	− 1	6	0	− 4	12	20
101	6	2	− 1	− 2	− 6	36	54
131	− 6	0	7	− 4	2	0	− 802
151	− 4	2	− 6	8	− 1	− 99	119
181	− 1	−10	2	6	2	0	338
191	0	− 9	− 2	2	8	78	266
*211	− 9	6	− 4	6	0	−156	192
241	0	6	−12	2	3	− 87	− 669
251	− 4	12	− 6	− 2	− 1	− 99	− 1801
271	8	− 6	− 9	0	6	42	− 1920
*281	−12	3	− 2	8	2	−150	− 1132
311	7	8	0	− 6	−10	114	872
331	− 6	0	2	− 9	12	180	1308
401	− 9	2	14	− 2	− 6	36	294
*421	12	2	2	−13	− 4	−138	6350
431	−16	0	2	6	7	− 35	− 1937
*461	− 4	− 4	16	− 9	0	−156	− 6108
491	15	− 4	−12	2	− 2	− 2	− 1244
*521	2	2	−18	7	6	−108	3530
541	0	− 9	8	12	−12	168	2796
571	−12	14	6	0	− 9	− 3	8895
601	− 4	2	14	3	−16	306	− 3346
631	− 1	0	12	6	−18	390	− 6252
641	−20	6	8	2	3	− 87	6051
661	−13	8	−10	14	0	−506	− 1398
*691	− 6	− 1	14	−16	8	− 72	−13084
701	16	2	− 6	3	−16	306	− 2266
751	− 4	−18	− 1	8	14	246	− 76
761	12	8	0	− 1	−20	484	− 8768
811	2	−12	−10	− 1	20	504	8672
821	3	14	− 4	−20	6	42	− 1080
*881	−22	− 2	3	8	12	30	− 6402
911	2	8	−20	− 6	15	189	− 6873
941	15	6	− 2	2	−22	588	−12034
971	− 2	24	− 4	−10	− 9	− 3	− 3585
*991	24	− 1	− 6	− 6	−12	18	2106

TABLE OF COEFFICIENTS OF THE PERIOD EQUATIONS.

p	P_2	P_3	P_4	P_5
11	− 4	− 3	3	1
31	− 12	− 21	1	5
41	− 16	5	21	− 9
61	− 24	− 17	41	− 13
71	− 28	37	25	1
101	− 40	93	− 21	− 17
131	− 52	− 89	109	193
151	− 60	− 12	784	128
181	− 72	− 123	223	− 49
191	− 76	− 359	− 437	− 155
211	− 84	− 59	1661	269
241	− 96	− 212	1232	512
251	−100	− 20	1504	1024
271	−108	− 401	− 13	845
281	−112	− 191	2257	967
311	−124	535	− 413	539
331	−132	− 887	− 1843	− 1027
401	−160	369	879	− 29
421	−168	219	3853	− 3517
431	−172	− 724	1824	1728
461	−184	− 129	4551	5419
491	−196	59	2019	1377
521	−208	− 771	4143	− 2085
541	−216	1147	− 805	− 2629
571	−228	868	3056	− 7552
601	−240	1755	− 3731	2399
631	−252	2095	− 4985	5069
641	−256	− 564	5328	− 5120
661	−264	− 185	16837	4851
691	−276	−1299	5329	15581
701	−280	2047	− 3791	1699
751	−300	−2313	− 3761	− 571
761	−304	2831	− 8925	8775
811	−324	−3471	−12431	−13603
821	−328	−1215	3573	2179
881	−352	−2361	4257	9967
911	−364	−2988	− 1392	9856
941	−376	3877	−13445	15271
971	−388	1476	8304	7168
991	−396	2101	8039	− 1819

I take the opportunity of noting the following corrections to Prof. Cayley's trisection table (A, p. 10),

p	
13	the coefficient of η^0 should be $+1$
37	$a, b, c = -10, -7, -8$
61	$b, c = -15, -12$
97	$b, c = -22, -23$

[*Postscript* (Added July, 1887).

Prof. Cayley remarks, "that $q\omega$ should be deducible from the product of two of the four prime factors of p as given in Reuschle's *Tafeln* (Berlin, 1875), p. 3. Thus, $p=11$, $f\omega \,.\, f\omega^2=(\omega^2+2\omega^4)(\omega^4+2\omega^3)$ which, subtracting $2(1+\omega+\omega^2+\omega^3+\omega^4)$, is $=2\omega^2-2\omega^4-\omega$, and, writing ω^3 for ω, this becomes $2\omega-2\omega^2-\omega^3$, which is the $q\omega$ of the memoir; but possibly in other cases the necessity of introducing complex units might interfere with the easy identification of the results."

The method of calculating $q\omega$ here suggested by Prof. Cayley is so easy in working, that it is important to show that it gives the desired result in all cases and without ambiguity.

Suppose $f\omega$ a prime factor of p; then we have

$$q\omega^i = u\omega \,.\, f\omega \,.\, f\omega^2,$$

where $u\omega, = u_0+u_1\omega+u_2\omega^2+u_4\omega^4+u_3\omega^3$, is a complex unit. Multiply each side of this equation by its conjugate (ω^{-1} written for ω). Then, since

$$q\omega^i \,.\, q\omega^{-i} = p = f\omega \,.\, f\omega^2 \,.\, f\omega^4 \,.\, f\omega^3,$$

we get

$$\begin{aligned} 1 &= u\omega \,.\, u\omega^{-1} \\ &= (u_0+u_1\omega+\ldots)(u_0+u_1\omega^{-1}+\ldots) \\ &= (u_0^2+u_1^2+u_2^2+u_4^2+u_3^2)+(u_1u_0+\ldots)\,\omega+\&c. \end{aligned}$$

Since u_0, u_1, &c. are real integers, it follows, from comparing the absolute terms, that one of $u^2=1$, and that all the rest vanish. The coefficient of each power of ω in the expansion of $u\omega \,.\, u\omega^{-1}$ then vanishes without further condition, so that

$$u\omega = \pm\omega^n, \quad (n=0, 1, 2, 3, 4).$$

Hence

$$q\omega^i = \pm\omega^n \,.\, f\omega \,.\, f\omega^2 + m\,(1+\omega+\omega^2+\omega^4+\omega^3) \quad \ldots\ldots\ldots(24).$$

It is to be proved that m, n and the $\pm$ can always be chosen, and that in one way only, so that the expression on the right of (24) may be identical with one of the four numbers, $q\omega^i$.

Note that the sum of the coordinates of $f\omega \,.\, f\omega^2$ is a square number (the square of the sum of the coordinates of $f\omega$), and it is therefore $\equiv \pm 1 \pmod 5$. Hence we can choose m, and a sign from the $\pm$ in (24), so as to get a complex integer, the sum of whose coordinates is -1. At each stage there is only one proper choice. (24) now takes the form

$$q\omega^i = \omega^n (a + b\omega + c\omega^2 + e\omega^4 + d\omega^3) \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

where a, b, c, d, e are known integers, such that

$$a + b + c + d + e = -1 \ldots\ldots\ldots\ldots\ldots\ldots (7').$$

Hence the residues of a, b, c, d, e to modulus 5 cannot be all equal nor all unequal, for in either case their sum would be $\equiv 0$. Two at least of these coordinates have, then, the same residue, say r; and all the essentially distinct sets of residues which are consistent with (7') are shown in the following Table (omitting the first column), where each row gives a set of residues,

$\frac{1}{2}\lambda$,	r,	r,	r,	r,	$r+4$,
$\frac{1}{2}\lambda+2$,	r,	r,	r,	$r+1$,	$r+3$,
$\frac{1}{2}\lambda+1$,	r,	r,	r,	$r+2$,	$r+2$,
$\frac{1}{2}\lambda$,	r,	r,	$r+1$,	$r+1$,	$r+2$,
$\frac{1}{2}\lambda-1$,	r,	r,	$r+2$,	$r+3$,	$r+4$.

But, since $(a+b\omega+\ldots)$ is one of two conjugate factors of p, its coordinates must satisfy the equations (*cf.* p. 217),

$$\left.\begin{aligned} ab+bc+cd+de+ea &= -\lambda \\ ac+ce+eb+bd+da &= -\lambda \\ a^2+b^2+c^2+d^2+e^2 &= 4\lambda+1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (11').$$

The third of these equations shows that the r of the first row of the above Table must be $\equiv 3\lambda$ (which $\equiv \frac{1}{2}\lambda \equiv$ the tens-figure in p), and similarly for each of the other rows r is the residue of the number written in the first column.

The first set of residues in the Table comprises four congruent to $\frac{1}{2}\lambda$, and a fifth congruent to $\frac{1}{2}\lambda - 1$. Since q_0 is known to be congruent to $\frac{1}{2}\lambda - 1$, we must take for q_0 that one of the coordinates a, b, ... which is congruent to $\frac{1}{2}\lambda - 1$; and n, in (25), must be such as to make this coordinate the absolute term. The determination of $q\omega^i$ is in this case complete.

The residues in the second, third, and fourth rows of Table include none that are congruent to $\frac{1}{2}\lambda-1$; so that there is no coordinate that can be taken for q_0. I find, however, that coordinates with these residues cannot occur on the right of (25), as they do not satisfy the first two equations (11′) in whatever way they are arranged.

In the case corresponding to the last row of the Table, there are two coordinates (call them a', e') congruent to $\frac{1}{2}\lambda-1$, and we have to decide which is the right one to take for q_0. Suppose it is a', and, when n is fixed accordingly, let $e'\omega^j$ be the term involving e', so that j is known. Now write $-j$ for the undetermined index i, in (25), and then change ω^{-j} into ω. Thus we get

$$q\omega = a'+b'\omega+c'\omega^2+e'\omega^4+d'\omega^3,$$

where a', b', c', d', e' are a, b, c, d, e in a different order, but still satisfying (7′), (11′); and $a'\equiv e'\equiv\frac{1}{2}\lambda-1$. The second (improper) form is easily found to be

$$\omega^4 q\omega^4 = e'+d'\omega+c'\omega^2+a'\omega^4+b'\omega^3;$$

and, to distinguish one from the other, we must have further information as to the coordinates b', c', d'. From (7′) and the second equation of (11′), we find

$$c'\equiv\tfrac{1}{2}\lambda+2,$$

but the equations become symmetrical with respect to b', d'. To complete the investigation, we use the equation

$$\sigma_2 = q^2+(q_1+q_4)(q_2+q_3),$$

which is

$$\sigma_2 = a^2+(b+e)(c+d)$$

(where the accents have been omitted as unnecessary). Now,

$$a\equiv e\equiv\tfrac{1}{2}\lambda-1,\quad c\equiv\tfrac{1}{2}\lambda+2,$$

so that, supposing

$$b\equiv\tfrac{1}{2}\lambda+1,\quad d\equiv\tfrac{1}{2}\lambda+3,$$

we get

$$\sigma_2\equiv-\lambda+1.$$

On the alternative hypothesis, viz.,

$$b\equiv\tfrac{1}{2}\lambda+3,\quad d\equiv\tfrac{1}{2}\lambda+1,$$

we should have

$$\sigma_2\equiv-\lambda+2.$$

But only the former congruence for σ_2 is consistent with P_4 being integral (see p. 226).

Hence $b\equiv\frac{1}{2}\lambda+1$, $d\equiv\frac{1}{2}\lambda+3$, and the proper form is characterised

by the condition

$$d-b, = q_3-q_1, \equiv 2 \pmod 5),$$

if q_4 is the coordinate congruent to q_0.

I note that the properties recorded in the concluding paragraphs of the paper as a result of observation, have now been proved to obtain generally.

As an example of the working, take

$$p = 31, \quad f\omega \,.\, f\omega^2 = (2\omega^2-\omega^4)(2\omega^4-\omega^3) \quad \text{(Reuschle, p. 3)}$$
$$= -2+4\omega+\omega^2-2\omega^3,$$

where sum of coordinates $= 1$. Therefore

$$q\omega^i = \omega^n (2-4\omega-\omega^2+2\omega^3),$$

so that $n = 0$, or 2; therefore

$$q\omega = 2- \omega+2\omega^4-4\omega^3, \qquad (n = 0,\ i = 2)$$

or

$$2-4\omega+2\omega^4- \omega^3, \qquad (n = 2,\ i = 3).$$

Of these, the former is right, since

$$q_3-q_1 = -4+1 = -3 \equiv 2.]$$

March 10th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Messrs. A. E. Hough Love, B.A., Fellow of St. John's College, Cambridge, and A. W. Cave, M.A., Oxford High School, were elected Members.

The following communications were made :—

A Metrical Property of Plane Curves: R. Lachlan.

Note on the Weierstrass Functions: A. G. Greenhill.

Second Paper on change of the Independent Variable, with applications to some Functions of the Reciprocant kind: C. Leudesdorf.

Note on Knots: the Treasurer.

The following presents were received:

"Proceedings of the Royal Society," Vol. XLI., No. 250.

"Educational Times," for March.

"Proceedings of the Physical Society," Vol. VIII., Pt. III., Jan., 1887.

"Hints for the Solution of Problems in the Third Edition of Solid Geometry," by Percival Frost, D.Sc., F.R.S.; 8vo, London, 1887.

Royal Irish Academy—Proceedings, "Science," Vol. IV., No. 5; "Polite Literature and Antiquities," Vol. II., No. 7. Transactions—"Science," Vol. XXVIII., Nos. 21 to 25; "Polite Literature and Antiquities," Vol. XXVII., Nos. 6 to 8. "Cunningham Memoirs," Nos. 2 and 3.

"Johns Hopkins University Circulars," Vol. VI., No. 55.

"Bulletin des Sciences Mathématiques," Feb., 1887.

"Annales de l'Ecole Polytechnique de Delft," Tome II., Liv. 3 and 4.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 1 and 2; Roma, 1887.

"Beiblätter zu den Annalen der Physik und Chemie," Index to B. X.

"Mitteilungen der Mathematischen Gesellschaft in Hamburg," No. 7.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 27 and 28.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," XL. to LIII.

"Berichte uber die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig." Mathematisch-Physische Classe, 1886. Supplement.

Second Paper on Change of the Independent Variable; with applications to some Functions of the Reciprocant kind. By C. LEUDESDORF, M.A.

[*Read March 10th*, 1887.]

In a previous paper (*Proceedings*, Vol. XVII., p. 329) I have shown how any rational integral homogeneous isobaric function of the differential coefficients of y with respect to x can, by means of the operator $e^{-V/t}$, be expressed as a similar function of the differential coefficients of x with respect to y. The variables x and y are supposed to be connected by one equation, and V is the annihilator of pure reciprocants. The objects of the present paper are as follows. Firstly, to investigate the more general problem of the change of the independent variable from x to z in a function of the differential coefficients of y and z with respect to x; the three variables being supposed connected by two equations. Secondly, to develop some of the properties of the differential operators to which the solution of the just mentioned problem leads. Thirdly, to discuss some of the systems of functions of the reciprocant kind which arise in connection with these operators. Some of these functions have, I find, been treated of by M. Halphen in a memoir "Sur les Invariants Différentiels des Courbes Gauches," published in the *Journal de l'Ecole Polytechnique*, XLVII^e^ Cahier, 1880; but his method of approaching the subject is quite

different from mine. I have made use of Mr. Elliott's method of symbolical representation for differential operators of the V class (see his paper, *Proceedings*, Vol. XVII., p. 142), as affording not only the most convenient notation for such operators, but also in many respects the simplest method of forming them.

Change of the Independent Variable.

1. Let x, y, z be three variables connected together by any two equations; denote $\frac{1}{n!}\frac{d^n x}{dz^n}$, $\frac{1}{n!}\frac{d^n y}{dz^n}$ by x_n, y_n respectively, and $\frac{1}{n!}\frac{d^n y}{dx^n}$, $\frac{1}{n!}\frac{d^n z}{dx^n}$ by Y_n, Z_n respectively, all the differential coefficients being of course *total*. We can easily express Y_1, Y_2, ... and Z_1, Z_2, ... in terms of x_1, x_2, ... and y_1, y_2, ...; in fact,

$$Y_1 = y_1 \qquad \div x_1,$$

$$Y_2 = x_1 y_2 - x_2 y_1 \qquad \div x_1^3,$$

$$Y_3 = x_1^2 y_3 - x_1(2y_2 x_2 + y_1 x_3) + 2x_2^2 y_1 \qquad \div x_1^5,$$

$$Y_4 = x_1^3 y_4 - x_1^2(3y_3 x_2 + 2y_2 x_3 + y_1 x_4) + 5x_1(y_2 x_2^2 + y_1 x_2 x_3) - 5x_2^3 y_1 \div x_1^7,$$

&c. &c.*; and

$$Z_1 = 1 \qquad \div x_1,$$

$$Z_2 = -x_2 \qquad \div x_1^3,$$

$$Z_3 = -x_1 x_3 + 2x_2^2 \qquad \div x_1^5,$$

$$Z_4 = -x_1^2 x_4 + 5x_1 x_2 x_3 - 5x_2^3 \qquad \div x_1^7,$$

&c. &c.*

It is seen at once that the expression for Y_n, in terms of x's and y's will in the first place involve y_n only once, in the term $\frac{y_n}{x_1^n}$; secondly,

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* It may here be remarked that it is not necessary, in order to obtain the above results, to perform the actual work of differentiation. The simplest method is to make use of the equations (1)—(4) of § 2. Thus, by substituting in (3) the values of ξ and η given by (1) and (2),

$$y_1\zeta + y_2\zeta^2 + y_3\zeta^3 + \ldots = Y_1(x_1\zeta + x_2\zeta^2 + x_3\zeta^3 + \ldots) + Y_2(x_1\zeta + x_2\zeta^2 + x_3\zeta^3 + \ldots)^2 + \&c.,$$

and by equating the coefficients of the successive powers of ζ on either side of this

that it must be homogeneous, and of degree $-(n-1)$, since the numerator will be of degree n, and the denominator is x_1^{2n-1}; thirdly, that it must be isobaric. If the weight of x_r, y_r be each taken as r, the weight of Y_n will be zero, since the numerator and denominator are each of weight $2n-1$. Lastly, it is noticed that it involves the y's only in the first degree, and that the coefficient of y_1 in Y_1, Y_2, Y_3, Y_4 is precisely equal to Z_1, Z_2, Z_3, Z_4. A proof that the coefficient of y_1 in Y_n is Z_n, is given in the next article.

2. Let x receive a finite increment ξ; and let the consequent increments of y and z be η and ζ respectively. By Taylor's theorem,

$$\xi = x_1\zeta + x_2\zeta^2 + x_3\zeta^3 + \dots \quad \dots\dots\dots\dots\dots\dots(1),$$

$$\eta = y_1\zeta + y_2\zeta^2 + y_3\zeta^3 + \dots \quad \dots\dots\dots\dots\dots\dots(2),$$

$$\eta = Y_1\xi + Y_2\xi^2 + Y_3\xi^3 + \dots \quad \dots\dots\dots\dots\dots\dots(3),$$

$$\zeta = Z_1\xi + Z_2\xi^2 + Z_3\xi^3 + \dots \quad \dots\dots\dots\dots\dots\dots(4).$$

Differentiating (3) for y_1,

$$\frac{dY_1}{dy_1}\xi + \frac{dY_2}{dy_1}\xi^2 + \frac{dY_3}{dy_1}\xi^3 + \dots = \frac{d\eta}{dy_1}$$

$$= \zeta \text{ by (2)}$$

$$= Z_1\xi + Z_2\xi^2 + Z_3\xi^3 + \dots \text{ by (4)},$$

so that, equating coefficients of ξ^n on each side,

$$\frac{dY_n}{dy_1} = Z_n.$$

3. Now let F be any function whatever of $Y_1, Y_2, \dots Y_n$ and $Z_1, Z_2, \dots Z_n$. It can be expressed as a function of $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots y_n$; when this has been done, suppose it to become Φ. With the same notation as in article 2,

$$y_1\zeta + y_2\zeta^2 + y_3\zeta^3 + \dots = \eta,$$

---

identity we obtain the values of Y_1, Y_2, Y_3, &c., in terms of x_1, x_2, x_3, &c., and y_1, y_2, y_3, &c. Again, from (1) and (4),

$$\xi = x_1(Z_1\xi + Z_2\xi^2 + Z_3\xi^3 + \dots) + x_2(Z_1\xi + Z_2\xi^2 + Z_3\xi^3 + \dots)^2 + \&c.,$$

and by equating the coefficients of the successive powers of ξ on either side of this identity we obtain the values of Z_1, Z_2, Z_3, &c., in terms of x_1, x_2, x_3, &c.

therefore $$\frac{dy_1}{dZ_1}\zeta+\frac{dy_2}{dZ_1}\zeta^2+\frac{dy_3}{dZ_1}\zeta^3+\ldots$$

$$+(y_1+2y_2\zeta+3y_3\zeta^2+\ldots)\frac{d\zeta}{dZ_1}=\frac{d\eta}{dZ_1}.$$

But $\frac{d\eta}{dZ_1}=0$, evidently; and $\frac{d\zeta}{dZ_1}=\xi$ by (4),

therefore $$\frac{dy_1}{dZ_1}\zeta+\frac{dy_2}{dZ_1}\zeta^2+\frac{dy_3}{dZ_1}\zeta^3+\ldots$$

$$=-(y_1+2y_2\zeta+3y_3\zeta^2+\ldots)\,\xi$$

$$=-(y_1+2y_2\zeta+3y_3\zeta^2+\ldots)(x_1\zeta+x_2\zeta^2+x_3\zeta^3+\ldots)\ldots\ldots\ldots\ldots(5)$$

by (1); so that, equating coefficients of the various powers of ζ on either side of this equation,

$\frac{dy_n}{dZ_1}$ = coefficient of ζ^n in the right-hand expression in (5).

Again, since $$x_1\zeta+x_2\zeta^2+x_3\zeta^3+\ldots=\xi,$$

$$\frac{dx_1}{dZ_1}\zeta+\frac{dx_2}{dZ_1}\zeta^2+\frac{dx_3}{dZ_1}\zeta^3+\ldots$$

$$=-(x_1+2x_2\zeta+3x_3\zeta^2+\ldots)\frac{d\zeta}{dZ_1}$$

$$=-(x_1+2x_2\zeta+3x_3\zeta^2+\ldots)(x_1\zeta+x_2\zeta^2+x_3\zeta^3+\ldots)\quad\ldots\ldots(6),$$

and, equating coefficients as before,

$\frac{dx_n}{dZ_1}$ = coefficient of ζ^n in the right-hand expression in (6).

Thus $$\frac{dF}{dZ_1}=\Sigma\frac{d\Phi}{dy_n}\frac{dy_n}{dZ_1}+\Sigma\frac{d\Phi}{dx_n}\frac{dx_n}{dZ_1}$$

$$=-(y_1+2y_2\zeta+3y_3\zeta^2+\ldots)(x_1\zeta+x_2\zeta^2+x_3\zeta^3+\ldots)$$

$$-(x_1+2x_2\zeta_1+3x_3\zeta_1^2+\ldots)(x_1\zeta_1+x_2\zeta_1^2+x_3\zeta_1^3+\ldots),$$

if only, after multiplying out, ζ^n be replaced by $\frac{d\Phi}{dy_n}$, and ζ_1^n by $\frac{d\Phi}{dx_n}$.

Therefore $$-\frac{dF}{dZ_1} = y_1x_1\frac{d\Phi}{dy_1} + (2y_2x_1+y_1x_2)\frac{d\Phi}{dy_2}$$

$$+(3y_3x_1+2y_2x_2+y_1x_3)\frac{d\Phi}{dy_3} + (4y_4x_1+3y_3x_2+2y_2x_3+y_1x_4)\frac{d\Phi}{dy_4} + \&c.$$

$$+x_1^2\frac{d\Phi}{dx_1} + 3x_1x_2\frac{d\Phi}{dx_2} + (4x_1x_3+2x_2^2)\frac{d\Phi}{dx_3} + (5x_1x_4+5x_2x_3)\frac{d\Phi}{dx_4} + \&c.$$

Of the expression on the right, let $W\Phi$ denote that part which does not involve either x_1 or y_1; then, rearranging the terms,

$$-\frac{dF}{dZ_1} = W\Phi + y_1\left(x_2\frac{d}{dy_2}+x_3\frac{d}{dy_3}+\ldots\right)\Phi + x_1\left(x_2\frac{d}{dx_2}+x_3\frac{d}{dx_3}+\ldots\right)\Phi$$

$$+x_1\left(x_1\frac{d}{dx_1}+2x_2\frac{d}{dx_2}+\ldots+y_1\frac{d}{dy_1}+2y_2\frac{d}{dy_2}+\ldots\right)\Phi$$

$$= W\Phi + y_1J\Phi + x_1\left(x_1\frac{d}{dx_1}+x_2\frac{d}{dx_2}+\ldots\right)\Phi - y_1x_1\frac{d\Phi}{dy_1} - x_1^2\frac{d\Phi}{dx_1}$$

$$+x_1\left(x_1\frac{d}{dx_1}+2x_2\frac{d}{dx_2}+\ldots+y_1\frac{d}{dy_1}+2y_2\frac{d}{dy_2}+\ldots\right)\Phi,$$

denoting by J the operator

$$x_1\frac{d}{dy_1}+x_2\frac{d}{dy_2}+x_3\frac{d}{dy_3}+\&c.$$

If now Φ is a function which is homogeneous (of degree i) in the x's, and isobaric (of total weight w) in the x's and y's *combined*,

$$-\frac{dF}{dZ_1} = W\Phi + y_1J\Phi + x_1(w+i)\Phi - y_1x_1\frac{d\Phi}{dy_1} - x_1^2\frac{d\Phi}{dx_1} \quad \ldots\ldots(7).$$

It will be noticed that the part of W which does not involve $y_1, y_2, \ldots$ is the operator V of the ordinary reciprocant theory.

4. In the particular case where $F = Y_n$, we have (Art. 1) $w = 0$, and $\iota+\iota' = -(n-1)$, if ι' be the degree of Y_n as far as the y's are concerned. Since evidently $\iota' = 1$, i must be equal to $-n$. Moreover, as F is (see Art. 3) a function of Y_1, Y_2, Y_3, ... and Z_1, Z_2, Z_3, ..., we have in this case $\frac{dF}{dZ_1} = 0$. And, since $y_1, y_2, y_3, \ldots y_n$ only enter in

the first degree in the expression for Y_n,

$$JY_n = \text{result of substituting } x\text{'s for the corresponding } y\text{'s throughout in this expression}$$

$$= \frac{d^n x}{dx^n}$$

$$= 0,$$

except when $n = 1$.

Accordingly, for all values of n greater than unity,

$$WY_n = nx_1 Y_n + x_1^2 \frac{dY_n}{dx_1} + x_1 y_1 \frac{dY_n}{dy_1}$$

$$= nx_1 Y_n + x_1^2 \frac{dY_n}{dx_1} + x_1 y_1 Z_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

from what has been proved at the end of Art. 2.

Let now, as a particular case, $y = z$; then Y_n will become Z_n; at the same time $y_1 = 1$, $y_2 = 0$, $y_3 = 0$, &c., so that

$$WZ_n = nx_1 Z_n + x_1^2 \frac{dZ_n}{dx_1} + x_1 Z_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

It will be noticed that this is the same formula as that given by me in result (11), p. 204, Vol. XVII., of the *Proceedings*; for, as has been seen, the operator W reduces to V when it acts upon anything which does not involve differential coefficients of y.

Subtracting y_1 times (9) from (8),

$$W(Y_n - y_1 Z_n) = nx_1 (Y_n - y_1 Z_n) + x_1^2 \frac{d}{dx_1} (Y_n - y_1 Z_n).$$

If, then, $$\Psi_n = x_1^{2n-1} (Y_n - y_1 Z_n),$$

$$W\left\{\Psi_n x_1^{-(2n-1)}\right\} = nx_1 \Psi_n x_1^{-(2n-1)} + x_1^2 \frac{d}{dx_1} \left\{\Psi_n x_1^{-(2n-1)}\right\},$$

therefore $$W\Psi_n = nx_1 \Psi_n + x_1^2 \frac{d\Psi_n}{dx_1} - (2n-1)\, x_1 \Psi_n$$

$$= x_1^2 \frac{d\Psi_n}{dx_1} - (n-1)\, x_1 \Psi_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

5. Suppose Ψ_n to be written in the form

$$A_0 x_1^{n-1} + A_1 x_1^{n-2} q + A_2 x_1^{n-3} q^2 + \ldots + A_{n-2} x_1 q^{n-2},$$

where A_0, A_1, &c. are functions which do not involve x_1 (A_0 is, in fact, y_n), and $q = 1$ has been inserted to make the expression homogeneous. Then

$$x_1 \frac{d\Psi_n}{dx_1} + q \frac{d\Psi_n}{dq} = (n-1)\,\Psi_n,$$

and by means of this, equation (10), the left-hand side of which must be multiplied by q to preserve homogeneity, takes the very simple form

$$W\Psi_n = -x_1 \frac{d\Psi_n}{dq} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

But $W\Psi_n = x_1^{n-1} WA_0 + x_1^{n-2} q WA_1 + x_1^{n-3} q^2 W^2 A_2 + \ldots + x_1 q^{n-2} W^{n-2} A_{n-2}$,

and $x_1 \dfrac{d\Psi_n}{dq} = x_1^{n-1} A_1 + 2x_1^{n-2} q A_2 + 3x_1^{n-3} q^2 A_3 + \ldots + (n-2)\, x_1 q^{n-3} A_{n-2}$.

Therefore, equating coefficients of the various powers of q,

$$\left.\begin{aligned} WA_0 &= -A_1 \\ WA_1 &= -2A_2 \\ WA_2 &= -3A_3, \text{ \&c.} \\ WA_{n-3} &= -(n-2)\, A_{n-2} \\ WA_{n-2} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and we may write

$$\Psi_n = x_1^{n-1} y_n - x_1^{n-2} q W y_n + \frac{1}{1.2} x_1^{n-3} q^2 W^2 y_n - \text{\&c.},$$

or, symbolically, $\Psi_n = x_1^{n-1} e^{-W/x_1} y_n$,

therefore $Y_n = y_1 Z_n + x_1^{-n} e^{-W/x_1} y_n$.

But $Z_n = -x_1^{-(2n-1)} x_1^{n-2} e^{-V/x_1} x_n$,

by equation (19), p. 208, Vol. XVII., of the *Proceedings*; and, as the effect of W on x_n is the same as that of V, we may write

$$Z_n = -x_1^{-(n+1)} e^{-W/x_1} x_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Therefore $Y_n = x_1^{-n} e^{-W/x_1} \left(y_n - \dfrac{y_1}{x_1} x_n \right) \ldots\ldots\ldots\ldots\ldots (14)$,

and we have now two formulæ, (14) and (13), which give the values of any differential coefficients of y and z with respect to x in terms of differential coefficients of y and x with respect to z.

6. The results just arrived at may now be extended. The method is precisely analogous to that adopted in my previous paper (*Proc.*, Vol. XVII., pp. 332, 333), and it is therefore not necessary to go through the work in detail. Referring to this paper, it will be seen that the formula (8) there given on p. 332 is the same, except for notation, as the formula (13) above; and that the formula (9) to which it leads is equivalent to the following, viz., that if f is any homogeneous and isobaric function of Z_2, Z_3, &c. of degree i and weight w (reckoning Z_r as of degree 1 and weight r), then

$$f(Z_2, Z_3, \ldots) = (-1)^i x_1^{-(w+i)} e^{-W/x_1} f(x_2, x_3, \ldots),$$

which may be written in the form

$$f(Z_2, Z_3, \ldots) = x_1^{-w} e^{-W/x_1} f\left(-\frac{x_2}{x_1}, -\frac{x_3}{x_1}, \ldots\right).$$

If the reasoning by which this last result is deduced be followed out, step for step, in the case of (14) above, we shall be led to the analogous result,

$$f(Y_2, Y_3, \ldots) = x_1^{-w} e^{-W/x_1} f\left(y_2 - \frac{y_1}{x_1} x_2,\ y_3 - \frac{y_1}{x_1} x_3,\ \ldots\right).$$

The last two formulæ may now be included in one, and we have the following theorem :—

If f be any homogeneous and isobaric function of Y_2, Y_3, ... and Z_2, Z_3, ... of total weight w, the degree of Y_r, Z_r being each reckoned as 1 *and their weights as r, then*

$$f(Y_2, Y_3, \ldots Y_m, Z_2, Z_3, \ldots Z_n)$$

$$= x_1^{-w} e^{-W/x_1} f\left(y_2 - \frac{y_1}{x_1} x_2,\ y_3 - \frac{y_1}{x_1} x_3,\ \ldots y_m - \frac{y_1}{x_1} x_m,\ -\frac{x_2}{x_1}, -\frac{x_3}{x_1}, \ldots -\frac{x_n}{x_1}\right)$$

.....(15).

Since, further, $Y_1 = \frac{y_1}{x_1}$ and $Z_1 = \frac{1}{x_1}$, we have here the means of calculating what any function of differential coefficients of y and z with respect to x becomes when expressed as a function of differential coefficients of y and x with respect to z.

7. It can, of course, be shown, by similar reasoning to the foregoing, that if

$$X'_n = \frac{1}{n!}\frac{d^n x}{dy^n} \quad \text{and} \quad Z'_n = \frac{1}{n!}\frac{d^n z}{dy^n},$$

and if f denote any homogeneous and isobaric function of $X'_2, X'_3, \ldots$ and $Z'_2, Z'_3, \ldots$ of weight w, reckoning X'_r, Z'_r as each of degree 1 and weight r, then

$$f(X'_2, X'_3, \ldots X'_m, Z'_2, Z'_3, \ldots Z'_n)$$

$$= y_1^{-w} e^{-W/y_1} f\left(x_2 - \frac{x_1}{y_1}y_2,\ x_3 - \frac{x_1}{y_1}y_3, \ldots x_m - \frac{x_1}{y_1}y_m,\ -\frac{y_2}{y_1},\ -\frac{y_3}{y_1}, \ldots -\frac{y_n}{y_1}\right) \quad \ldots\ldots\ldots\ldots\ldots(16),$$

where W is the operator

$$2x_2y_2\frac{d}{dx_3} + (3x_3y_2 + 2x_2y_3)\frac{d}{dx_4} + (4x_4y_2 + 3x_3y_3 + 2x_2y_4)\frac{d}{dx_5} + \ldots$$

$$+ 2y_2^2\frac{d}{dy_3} + 5y_2y_3\frac{d}{dy_4} + (6y_2y_4 + 3y_3^2)\frac{d}{dy_5} + \ldots,$$

similar to W, but with x and y interchanged.

In the course of the work an operator will be arrived at similar to J, but with x and y interchanged, viz.,

$$y_1\frac{d}{dx_1} + y_2\frac{d}{dx_2} + y_3\frac{d}{dx_3} + \ldots;$$

let this be denoted by J'.

Relations between the Operators W, W', J, J', $\frac{d}{dz}$.

8. The above operators are connected by some remarkable relations, which I proceed now to investigate.

For convenience, let W be divided into two parts, W_1 and V, so that

$$W_1 = 2y_2x_2\frac{d}{dy_3} + (3y_3x_2 + 2y_2x_3)\frac{d}{dy_4} + \ldots,$$

$$V = 2x_2^2\frac{d}{dx_3} + 5x_2x_3\frac{d}{dx_4} + \ldots.$$

Then, if f be a function of the x's of degree i and weight w, it is well known that

$$\left(V\frac{d}{dz} - \frac{d}{dz}V\right)f = 2x_2(w+i)f.$$

It may be worth while to give a proof of this, founded on the symbolical expressions for the operators.

We may write, symbolically,

$$\frac{d}{dz} = 2x_2\xi + 3x_3\xi^2 + 4x_4\xi^3 + \ldots,$$

$$V = (2x_2\xi + 3x_3\xi^2 + \ldots)(x_2\xi^2 + x_3\xi^3 + \ldots),$$

with the understanding that ξ^r is to be replaced by $\frac{d}{dx_r}$.

If, then, the square brackets denote the part which is obtained by mere multiplication of the symbols, and which is the same for $V\frac{d}{dz}$ and for $\frac{d}{dz}V$, we have

$$V\frac{d}{dz} - \left[V\frac{d}{dz}\right] = \text{effect of writing } 3\xi^3 \text{ for } \xi^5,\ 4\xi^3 \text{ for } \xi^4, \text{ \&c. in the expression for } V$$

$$= \frac{dV}{d\xi},$$

$$\frac{d}{dz}V - \left[\frac{d}{dz}V\right] = 3x_3\frac{dV}{dx_2} + 4x_4\frac{dV}{dx_3} + 5x_5\frac{dV}{dx_4} + \ldots$$

$$= \frac{d}{d\xi}\left\{(3x_3\xi^2 + 4x_4\xi^3 + \ldots)(x_2\xi^2 + x_3\xi^3 + \ldots)\right\}$$

$$= \frac{d}{d\xi}\left\{V - 2x_2\xi\,(x_2\xi^2 + x_3\xi^3 + \ldots)\right\}$$

$$= \frac{dV}{d\xi} - 2x_2\,(3x_2\xi^2 + 4x_3\xi^3 + \ldots).$$

Therefore, by subtraction,

$$\left(V\frac{d}{dz} - \frac{d}{dz}V\right)f = 2x_2\left(3x_2\frac{d}{dx_2} + 4x_3\frac{d}{dx_3} + \ldots\right)f$$

$$= 2x_2\,(w+i)\,f.$$

And if f denote any function which involves both x's and y's, and which is isobaric (combined weight w) in them *combined*; then, if it is at the same time homogeneous (of degree i) in the x's, a formula

similar to the one just proved will still hold, viz.,

$$\left(V\frac{d}{dz}-\frac{d}{dz}V\right)f=2x_2\left(2x_2\frac{d}{dx_3}+3x_3\frac{d}{dx_4}+\ldots\right)f+2x_2if\ldots(17),$$

for V acts only on the x-part of f.

We can now prove that, *if f is a function of the x's and y's of this kind, then* $(W\delta_z-\delta_z W)f=2x_2(w+i)f+2y_2Jf$.

For we can write, symbolically,

$$\frac{d}{dz}=2x_2\xi+3x_3\xi^2+4x_4\xi^3+\ldots+2y_2\eta+3y_3\eta^2+4y_4\eta^3+\ldots,$$

$$W_1=(2y_2\eta+3y_3\eta^2+4y_4\eta^3+\ldots)(x_2\eta^2+x_3\eta^3+x_4\eta^4+\ldots),$$

with the understanding that ξ^r, η^r are to be replaced by $\frac{d}{dx_r}$, $\frac{d}{dy_r}$, respectively. Therefore

$$W_1\frac{d}{dz}-\left[W_1\frac{d}{dz}\right]=\text{result of writing } 3\eta^2 \text{ for } \eta^3,\ 4\eta^3 \text{ for } \eta^4, \text{ \&c. in the expression for } W_1$$

$$=\frac{dW_1}{d\eta},$$

$$\frac{d}{dz}W_1-\left[\frac{d}{dz}W_1\right]=\text{result of writing } 3x_3 \text{ for } x_2,\ 4x_4 \text{ for } x_3, \text{ \&c. in } W_1$$

$$+\text{result of writing } 3y_3 \text{ for } y_2,\ 4y_4 \text{ for } y_3, \text{ \&c. in } W_1$$

$$=(2y_2\eta+3y_3\eta^2+\ldots)(3x_3\eta^2+4x_4\eta^3+\ldots)$$

$$+(2\,.\,3y_2\eta+3\,.\,4y_3\eta^2+\ldots)(x_2\eta^2+x_3\eta^3+\ldots)$$

$$=\frac{dW_1}{d\eta}-2x_2\eta(2y_2\eta+3y_3\eta^2+\ldots)-2y_2(x_2\eta^2+x_3\eta^3+\ldots).$$

Therefore, by subtraction,

$$\left(W_1\frac{d}{dz}-\frac{d}{dz}W_1\right)f$$

$$=2x_2\left(2y_2\frac{d}{dy_2}+3y_3\frac{d}{dy_3}+\ldots\right)f+2y_2\left(x_2\frac{d}{dy_2}+x_3\frac{d}{dy_3}+\ldots\right)f$$

$$=2x_2\left(2y_2\frac{d}{dy_2}+3y_3\frac{d}{dy_3}+\ldots\right)+2y_2Jf\quad\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

If then (17) and (18) be added together, we have the formula

$$(W\delta_z-\delta_z W)f=2x_2(w+i)f+2y_2Jf\quad\ldots\ldots\ldots\ldots(19).$$

And, by a similar method, the analogous result

$$(W'\delta_z - \delta_z W')f = 2y_2(w+i')f + 2x_2 J'f \quad \ldots\ldots\ldots\ldots (20)$$

may be demonstrated for a function f which is homogeneous (of degree i') in the y's, and isobaric (of weight w) in x's and y's *combined.*

9. *The operators*

$$J = x_1 \frac{d}{dy_1} + x_2 \frac{d}{dy_2} + x_3 \frac{d}{dy_3} + \ldots$$

and W are commutative, the quantity operated on being any function of the x's and y's.

For, since the y's occur in W only in the first degree,

$JW - [JW]$ = result of substituting x's for the corresponding y's in W

$= V$, except that $\frac{d}{dx_r}$ is replaced by $\frac{d}{dy_r}$ for all values of r.

And, since the x's occur in J only in the first degree,

$WJ - [WJ]$ = result of substituting $\frac{d}{dy_r}$ for $\frac{d}{dx_r}$ in V for all values of r.

Therefore, by subtraction,

$$JW - WJ = 0,$$

or

$$JW = WJ \quad \ldots\ldots\ldots\ldots (21).$$

Similarly it appears that the operators

$$J' = y_1 \frac{d}{dx_1} + y_2 \frac{d}{dx_2} + y_3 \frac{d}{dx_3} + \ldots$$

and W' are commutative; or that

$$J'W' = W'J' \quad \ldots\ldots\ldots\ldots (22),$$

the operand being any function of the x's and y's.

10. *The operators J, J', W, W' are further connected by the formulæ*

$$\left.\begin{aligned} JW' - W'J &= W \\ J'W - WJ' &= W' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (23).$$

To prove the first of these, write symbolically,

$$J = x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots$$

$$W' = (2x_2\xi + 3x_3\xi^2 + 4x_4\xi^3 + \ldots)(y_2\xi^2 + y_3\xi^3 + y_4\xi^4 + \ldots)$$
$$+ (2y_2\eta + 3y_3\eta^2 + 4y_4\eta^3 + \ldots)(y_2\eta^2 + y_3\eta^3 + y_4\eta^4 + \ldots)$$

where ξ^r, η^r, as before, are to be replaced by $\frac{d}{dx_r}$, $\frac{d}{dy_r}$ respectively.

Then $JW' - [JW']$ = result of replacing all the y's in the expression for W' by the corresponding x's

$$= (2x_2\xi + 3x_3\xi^2 + \ldots)(x_2\xi^2 + x_3\xi^3 + \ldots)$$
$$+ (2x_2\eta + 3x_3\eta^2 + \ldots)(y_2\eta^2 + y_3\eta^3 + \ldots)$$
$$+ (2y_2\eta + 3y_3\eta^2 + \ldots)(x_2\eta^2 + x_3\eta^3 + \ldots)$$
$$= W + (2x_2\eta + 3x_3\eta^2 + \ldots)(y_2\eta^2 + y_3\eta^3 + \ldots),$$

$W'J - [W'J]$ = result of replacing ξ^r in the expression for W' by η^r, and η^r by zero, for all values of r

$$= (2x_2\eta + 3x_3\eta^2 + \ldots)(y_2\eta^2 + y_3\eta^3 + \ldots).$$

Therefore, by subtraction,

$$JW' - W'J = W,$$

the operand being *any* function of $x_2, x_3, \ldots$, and $y_2, y_3, \ldots$; and in a similar manner the second of the formulæ (23) may be demonstrated.

11. *The following formula is also true, viz.*,

$$(WW' - W'W)\,\delta_z = \delta_z\,(WW' - W'W),$$

the operand being a function of $x_2, x_3, \ldots$ and $y_2, y_3, \ldots$ which is homogeneous both as regards the x's and as regards the y's, and which is isobaric in the x's and y's combined. Suppose f such a function; let i be its degree as far as the x's are concerned, and i' its degree as far as the y's are concerned, and let w be its total weight, *i.e.*, its weight in x's and y's combined.

Since, by (19),

$$(W\delta_z - \delta_z W)f = 2x_2\,(w+i)f + 2y_2 Jf,$$

therefore $$W'\,(W\delta_z - \delta_z W)f = 2x_2\,(w+i)\,W'f + 2y_2 W'Jf \quad \ldots\ldots\ldots (24).$$

Now the effect of W on f is evidently to increase its weight and its x-degree each by unity, while leaving its y-degree unaltered. Writing then Wf in place of f, in (20),

$$(W'\delta_z - \delta_z W')\, Wf = 2y_2\,(w+i''+1)\, Wf + 2x_2 J' Wf,$$

adding which to (24), we have

$$(W'W\delta_z - \delta_z W'W)\, f$$

$$= 2x_2\,\{(w+i)\, W'f + J'Wf\} + 2y_2\,\{(w+i''+1)\, Wf + W'Jf\} \ldots (25).$$

Again, acting with W on (20),

$$W(W'\delta_z - \delta_z W')\, f = 2y_2\,(w+i'')\, Wf + 2x_2 WJ'f \ldots\ldots\ldots (26).$$

Writing $W'f$ in place of f in (19), and observing that the effect of W' on f is to increase its weight and its y-degree each by unity, while leaving its x-degree unchanged,

$$(W\delta_z - \delta_z W)\, W'f = 2x_2\,(w+i+1)\, W'f + 2y_2 JW'f,$$

adding which to (26), we have

$$(WW'\delta_z - \delta_z WW')\, f$$

$$= 2x_2\,\{(w+i+1)\, W'f + WJ'f\} + 2y_2\,\{(w+i'')\, Wf + JW'f\} \ldots (27).$$

Finally, subtracting (25) from (27),

$$(WW' - W'W)\,\delta_z f - \delta_z\,(WW' - W'W)\, f$$

$$= 2x_2\,(W' + WJ' - J'W)\, f + 2y_2\,(JW' - W'J - W)\, f$$

$$= 0, \text{ by (23)};$$

so that $$(WW' - W'W)\,\delta_z f = \delta_z\,(WW' - W'W)\, f \ldots\ldots\ldots (28).$$

12. It can now be shown that W and W' *are commutative when they act on any function* ϕ *of* $x_2, x_3, \ldots$ *and* $y_2, y_3, \ldots$.

Suppose that $$(WW' - W'W)\, x_n = 0 \ldots\ldots\ldots\ldots\ldots (29),$$

for some particular value of n. Then, by (28),

$$(WW' - W'W)\, x_{n+1} = \frac{1}{n+1}\,\delta_z\,(WW' - W'W)\, x_n = 0.$$

But $$(WW' - W'W)\, x_4 = W\,(3y_3x_2 + 2y_2x_3) - W'\,(5x_2x_3)$$

$$= 6x_2^2y_2 + 4x_2^2y_2 - 10x_2^2y_2 = 0;$$

therefore, by induction, (29) is true for all values of n; and it can similarly be proved that

$$(WW' - W'W)\, y_n = 0,$$

for all values of n. Therefore

$$(WW' - W'W)\, \phi\,(x_2, x_3, \ldots\ y_2, y_3, \ldots)$$

$$= \Sigma \frac{d\phi}{dx_n} (WW' - W'W)\, x_n + \Sigma \frac{d\phi}{dy_n} (WW' - W'W)\, y_n = 0,$$

so that

$$WW'\phi = W'W\phi.$$

Applications of the foregoing to functions of the Reciprocant kind.

13. From (15) it is evident that $Wf = 0$ is the necessary and sufficient condition that a homogeneous and isobaric function of $Y_2, Y_3, \ldots Z_2, Z_3, \ldots$ should remain unaltered (except for some power of x_1 and a possible change of sign) when Y_n is replaced by $x_n y_1 - y_n x_1$ and Z_n by x_n. And similarly, from (16), it is seen that $W'f = 0$ is the necessary and sufficient condition that a similar function of $X'_2, X'_3, \ldots Z'_2, Z'_3, \ldots$ should remain unaltered (except for some power of y_1 and a possible change of sign) when X'_n is replaced by $y_n x_1 - x_n y_1$ and Z'_n by y_n. Referring to a previous paper of mine (*Proc.*, Vol. XVII., p. 197), it will be seen that the equations (12) of § 5 above are precisely analogous to equations (18), p. 208 of that paper, Ψ_n and W taking the places of Y_{n+1} and V there. Exactly then as proved on pp. 209, 210 of that paper, it can be shown that, if

$$x_1^{-1}\Psi_3 = x_1^4 (Y_3 - y_1 Z_3) = x_1 y_3 - 2q y_2 x_2,$$

$$x_1^{-1}\Psi_4 = x_1^6 (Y_4 - y_1 Z_4) = x_1^2 y_4 - x_1 q\,(3y_3 x_2 + 2y_2 x_3) + 5q^2 y_2 x_2^3,$$

$$x_1^{-1}\Psi_5 = x_1^8 (Y_5 - y_1 Z_5) = x_1^3 y_5 - x_1^2 q\,(4y_4 x_2 + 3y_3 x_3 + 2y_2 x_4)$$
$$+ x_1 q^2 (9y_3 x_2^2 + 12 y_2 x_2 x_3) - 14 q^3 y_2 x_2^3, \text{ \&c. \&c.},$$

($q = 1$ being inserted for the sake of homogeneity,) then any q-seminvariant of any of these quantics, or of any system formed by a number of them, is a function of the first class mentioned above; *i.e.*, it has the property that it remains unchanged when Y_n, Z_n are replaced by $x_n y_1 - x_1 y_n$, x_n respectively, and conversely.

For example, the discriminant of the second quantic is

$$(3y_3 x_2 + 2y_2 x_3)^2 - 20 y_2 y_4 x_2^2;$$

the resultant of the first and second is

$$x_2 y_2 (4x_2 y_2 y_4 - x_2 y_3^2 - 4x_3 y_2 y_3).$$

Accordingly, the functions

$$(3Y_3 Z_2 + 2Y_2 Z_3)^2 - 20 Y_2 Y_4 Z_2^2,$$

$$4Z_2 Y_2 Y_4 - Z_2 Y_3^2 - 4Z_3 Y_2 Y_3,$$

are such that, if $y_2 - \frac{y_1}{x_1} x_2$, &c. be written for y_2, &c., and $-\frac{x_2}{x_1}$, &c. for x_2, &c., they will be unaltered, except for some power of x_1 and a possible change of sign.

Since functions such as these will evidently be homogeneous both as regards the Y's and the Z's, we may say that they have the property of persistence when $Y_2, Y_3, \ldots$ are replaced by $y_2 - \frac{y_1}{x_1} x_2$, $y_3 - \frac{y_1}{x_1} x_3$, ... and $Z_2, Z_3, \ldots$ by $x_2, x_3, \ldots$. And, similarly, the functions which are annihilated by W' possess the property of persistence when $X_2', X_3', \ldots$ are replaced by $x_2 - \frac{x_1}{y_1} y_2$, $x_3 - \frac{x_1}{y_1} y_3$, ... and $Z_2', Z_3', \ldots$ by $y_2, y_3, \ldots$.

14. Considering the function

$$4Z_2 Y_2 Y_4 - Z_2 Y_3^2 - 4Z_3 Y_2 Y_3,$$

we see that W, acting on

$$4x_2 (y_2 - \lambda x_2)(y_4 - \lambda x_4) - x_2 (y_3 - \lambda x_3)^2 - 4x_3 (y_2 - \lambda x_2)(y_3 - \lambda x_3)$$

$\left(\text{where } \lambda = \frac{y_1}{x_1}\right)$, must give the result zero; that is, since W does not act on λ,

$$W (4x_2 y_2 y_4 - x_2 y_3^2 - 4x_3 y_2 y_3) + 2\lambda W (2x_2^2 y_4 + 2x_2 x_4 y_2 - x_2 x_3 y_3 - 2x_3^2 y_2) + \lambda^2 W (4x_2^2 x_4 - 5x_2 x_3^2) = 0.$$

It follows that the effect of W on each of the three quantities within the brackets must be zero. This result is evidently general, and may be thus stated:—

If f be a rational integral homogeneous isobaric function of $x_2, x_3, \ldots$ and $y_2, y_3, \ldots$ such that $Wf = 0$, then, if $y_2 - \lambda x_2$, $y_3 - \lambda x_3$, &c. be written in place of $y_2, y_3, \ldots$ and the result be

$$f + \lambda f_1 + \lambda^2 f_2 + \ldots + \lambda^n f_n,$$

then $Wf_1 = 0$, $Wf_2 = 0$, &c.; *i.e.*, f_1, f_2, &c. are themselves functions which have the same property as f as regards persistence.

It is clear that f_n is equal to the result of writing x for y throughout in f; and, as f_n involves only x's, it must be an ordinary pure reciprocant. Similar remarks apply, *mutatis mutandis*, to the second class of functions, those namely which are annihilated by W.

15. Of much greater interest, however, are the functions which possess the property of persistence when Z_n is replaced by x_n, and Y_n not by $y_n - \frac{y_1}{x_1} x_n$, but by y_n, for all values of n; *i.e.*, those functions in which x and z may be interchanged without altering them (except for a power of x_1 and a possible change of sign). These might be called *semi-reciprocants*, as they persist when linear functions of x and z are written in place of x and z, but y is kept unchanged. There will, of course, be a second set of semi-reciprocants which persist when Z'_n is replaced by y_n and X'_n by x_n for all values of n; *i.e.*, when for y and z are written linear functions of these, and x is kept unchanged. For convenience, and more particularly to distinguish them from each other and from the reciprocants considered below in § 22, I call these two classes of functions (xz, y) and (yz, x) *reciprocants* respectively. The ordinary or (xy) reciprocants will be included among each of these. They are, in fact, (xz, y) reciprocants, in which y does not occur at all, or (yz, x) reciprocants in which x does not occur at all. I confine myself to the (xz, y) semireciprocants, the case of the (yz, x) semireciprocants being in all respects analogous. It may be worth while to remark that there will be a corresponding theory relating to the general case of "reciprocants" which are functions of any number n of variables x, y, z, v, w, &c. There will be the "partial" reciprocants which persist when for some m of these variables are substituted linear functions of themselves, and the remaining $n-m$ are kept unaltered. And there will be the "total" reciprocants, for which $m = n$, which persist when linear substitutions are simultaneously impressed on all the variables, and which are analogous to the reciprocants considered below in § 22. And among these there will be a sub-class which are the analogues of those of § 27 below, remaining unchanged when all the variables are altered by homographic substitutions.

16. Much of what is proved in my previous paper (*Proceedings*, Vol. XVII., pp. 204, *seqq.*) applies *mutatis mutandis* to (xz, y) semireciprocants. For example,

$$\begin{aligned} Y_n \pm x_1^{\lambda} y_n &= Y_n \pm Z_1^{-\lambda} y_n \\ &= \pm Z_1^{-\lambda} (y_n \pm Z_1^{\lambda} Y_n), \end{aligned}$$

so that, if we *add* y_n multiplied by any power of x_1 to the expression for Y_n, as given in § 1, we shall have an (xz, y) mixed semireciprocant of *positive* character; and, if we *subtract* the same one from the other, we shall have a similar semireciprocant, but of *negative* character.

Thus,
$$(1+x_1)\,y_2-x_2y_1,$$
$$x_1\,(2x_2y_2+x_3y_1)-2x_2^2y_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

are semireciprocants of positive and negative character respectively. It may in fact be verified easily by actual substitution that they are respectively equal to
$$x_1^3\left[(1+Z_1)\,Y_2-Z_2Y_1\right],$$
and
$$-x_1^7\left[Z_1\,(2Z_2Y_2+Z_3Y_1)-2Z_2^2Y_1\right].$$

So, again, $2x_1y_2-x_2y_1$ is a semireciprocant, and is equal to
$$x_1^4\left[2Z_1Y_2-Z_2Y_1\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

It is unnecessary to multiply examples of this kind. If we choose λ equal to $n-1$, we obtain the most important class of these mixed semireciprocants, viz., the homogeneous ones. They form the two fundamental series,

$$P_2 = 2x_1y_2-x_2y_1,$$
$$P_3 = 2x_1^2y_3-x_1\,(2y_2x_2+y_1x_3)+2x^3y_1,$$
$$P_4 = 2x_1^3y_4-x_1^2\,(3y_3x_2+2y_2x_3+y_1x_1)+5x_1\,(y_2x_2^2+y_1x_2x_3)-5x_2^3y_1,\ \&c.,$$

all of positive character; and

$$N_3 = x_1\,(2y_2x_2+y_1x_3)-2x_2^2y_1,$$
$$N_4 = x_1^2\,(3y_3x_2+2y_2x_3+y_1x_4)-5x_1\,(y_2x_2^2+y_1x_2x_3)+5x_2^3y_1,\ \&c.,$$

all of negative character.

Since
$$Y_1=\frac{y_1}{x_1},\quad Y_2=\frac{y_2}{x_1^2}+\ldots,\quad Y_3=\frac{y_3}{x_1^3}+\ldots\ \&c.,$$

and
$$Z_1=\frac{1}{x_1},\quad Z_2=-\frac{x_2}{x_1^3}-\ldots,\quad Z_3=-\frac{x_3}{x_1^4}-\ldots\ \&c.,$$

it can easily be seen that, when a homogeneous semireciprocant of the (xz, y) class is transformed from a function of the Y's and Z's into a function of the x's and y's, it will take up as a factor x_1 raised to the

power $w+i$ (with the notation of §11). Thus the characteristic or *index* of such a semireciprocant is equal to its total weight plus its x-degree. Similarly the index of a homogeneous (yz, x) semireciprocant is $w+i'$, that is, its total weight plus its y-degree.

As another instance of a result proved for ordinary reciprocants which has its analogue in the case of these semireciprocants, may be cited that of (32), p. 213. It will clearly be true that the operator

$$(x_1^2 \pm x_1^k)\frac{d}{dx_1} - W$$

(where k is any integer), acting on a mixed homogeneous (xz, y) semireciprocant, will generate another of the same kind; and a similar theorem, with y_1 and W' in place of x_1 and W, will be true of (yz, x) semireciprocants.

17. I proceed now to the consideration of *pure* (xz, y) semireciprocants; such, namely, as do not involve either x_1 or y_1; these will be found to satisfy two independent differential equations. Let F be such a function, expressed in terms of Y's and Z's, and let its transform in terms of x's and y's be $x_1^{-(w+i)}\Phi$. It can be shown without difficulty that Φ must be homogeneous and isobaric both in x's and in y's. Since, moreover, Φ does not involve either x_1 or y_1, equation (7) of Art. 3 reduces in this case simply to

$$-\frac{dF}{dZ_1} = W\Phi + y_1 J\Phi.$$

Now, F does not involve Z_1, so that the right-hand member of this equation must vanish. But since it consists of two terms, one of which does, and the other does not, involve y_1, it is clear that each of these must separately vanish. Therefore

$$J\Phi = 0 \quad \text{and} \quad W\Phi = 0$$

are the necessary conditions that Φ be an (xz, y) semireciprocant.

18. It can further be shown that these two conditions are sufficient as well as necessary. For suppose f a homogeneous isobaric function such that it satisfies these two conditions. Referring back to equation (15), Art. 6, which will of course hold in this case, let the right-hand expression, which may be written

$$(-1)^i x_1^{-(w+i)} e^{-W/x_1} f(y_2 - \lambda x_2, y_3 - \lambda x_3, \ldots, x_2, x_3, \ldots),$$

$\left(\text{where } \lambda = \frac{y_1}{x_1}\right)$, be expanded by Taylor's theorem in powers of λ. Then, since $Jf = 0$, the expansion will reduce to its first term simply, since the coefficients of the successive powers of λ will all vanish. Thus,

$$f(Y_2, Y_3, \ldots, Z_2, Z_3, \ldots) = (-1)^i x_1^{-(w+i)}\, e^{-W/x_1} f(y_2, y_3, \ldots, x_2, x_3, \ldots)$$
$$= (-1)^i x_1^{-(w+i)} f(y_2, y_3, \ldots, x_2, x_3, \ldots),$$

since $Wf = 0$ and W^2f, W^3f, &c., also vanish. This shows that f must be an (xz, y) semireciprocant.

Similarly, the necessary and sufficient conditions that a function Φ of $x_2, x_3, \ldots y_2, y_3, \ldots$ should be a (yz, x) semireciprocant are

$$J'\Phi = 0 \quad \text{and} \quad W'\Phi = 0.$$

19. At this point it is convenient to adopt a different notation. Let

$$y_3x_2 - x_3y_2 = a_3,$$

$$y_4x_2 - x_4y_2 = 2a_4,$$

$$y_5x_2 - x_5y_2 = 3a_5, \text{ \&c.,}$$

then it is clear that any function Φ of the x's and y's can, by successive substitutions, be expressed as a function of the x's, the a's, and y_2. Considering the particular case where the function is an (xz, y) semireciprocant,

$$J\Phi = \frac{d\Phi}{da_3} Ja_3 + \frac{d\Phi}{da_4} Ja_4 + \ldots + \frac{d\Phi}{dy_2} Jy_2.$$

But, evidently $Ja_3 = 0$, $Ja_4 = 0$, &c., &c., and $Jy_2 = x_2$; therefore, since $J\Phi = 0$, it follows that

$$x_2 \frac{d\Phi}{dy_2} = 0,$$

i.e., Φ does not involve y_2. Accordingly, any (xz, y) semireciprocant can be expressed in terms of the x's and the a's; and it will evidently be homogeneous and isobaric in these new variables. And, conversely, any function of the x's and the a's which is homogeneous and isobaric in them will be annihilated by J; and if, further, it be annihilated by W,

it will be an (xz, y) semireciprocant. Since

$$Wa_3 = 0,$$

$$Wa_4 = 1.3\,x_2 a_3,$$

$$Wa_5 = 2.4\,x_2 a_4 + 1.3\,x_3 a_3,$$

$$Wa_6 = 3.5\,x_2 a_5 + 2.4\,x_3 a_4 + 1.3\,x_4 a_3, \text{ \&c.},$$

the operator W, when expressed in terms of the new variables, takes the form

$$\tfrac{1}{2}(1.3\,x_2 a_3)\frac{d}{da_4} + \tfrac{1}{3}(2.4 x_2 a_4 + 1.3 x_3 a_3)\frac{d}{da_5} + \ldots$$

$$+ 2x_2^2 \frac{d}{dx_3} + 5x_2 x_3 \frac{d}{dx_4} + \ldots$$

The formula (15) will take the convenient form

$$f(A_3, A_4, \ldots A_m, Z_2, Z_3, \ldots Z_n)$$

$$= (-1)^i x_1^{-(w+i)} e^{-W|x_1} f(a_3, a_4, \ldots a_m, x_2, x_3, \ldots x_n),$$

where

$$A_3 = Y_3 Z_2 - Y_2 Z_3,$$

$$2A_4 = Y_4 Z_2 - Y_2 Z_4,$$

and so on. It must be remembered that $w = 5, 6, 7 \ldots$ for a_3, a_4, a_5, ... &c., and that $i = 1$ in the case of all of them.

If, instead of a_3, a_4, a_5, ... we write a, b, c, ..., and instead of x_2, x_3, x_4, ... we write a', b', c', ..., then W becomes

$$\left.\begin{aligned}\tfrac{1}{2}(1.3aa')\,\delta_b + \tfrac{1}{3}(1.3ab' + 2.4a'b)\,\delta_c + \tfrac{1}{4}(1.3ac' + 2.4bb' + 3.5ca')\,\delta_d + \ldots \\ + 2a'^2\delta_{b'} + 5a'b'\delta_{c'} + (6a'c' + 3b'^2)\,\delta_{d'} + \ldots\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(32).$$

20. The operator W' cannot of course be expressed in terms of a, b, c, ... and a', b', c', ...; but $x_2 W' - y_2 W$ can, as follows:—

From the value of W' given in § 7, it is seen that

$$W'a = 0, \quad 2W'b = 1.3\,y_2 a, \quad 3W'c = 1.3\,y_3 a + 2.4\,y_2 b,$$

$$4W'd = 1.3\,y_4 a + 2.4\,y_3 b + 3.5\,y_2 c, \text{ \&c.};$$

$$W'a' = 0, \quad W'b' = 2y_2 a', \quad W'c' = 2y_3 a' + 3y_2 b',$$

$$W'd' = 2y_4 a' + 3y_3 b' + 4y_2 c', \text{ \&c.};$$

therefore

$$W'=\tfrac{1}{2}(1.3y_2a)\delta_b+\tfrac{1}{3}(1.3y_3a+2.4y_2b)\delta_c+\tfrac{1}{4}(1.3y_4a+2.4y_3b+3.5y_2c)\delta_d+\dots$$
$$+2y_2a'\delta_{b'}+(2y_3a'+3y_2b')\,\delta_{c'}+(2y_4a'+3y_3b'+4y_2c')\,\delta_{d'}+\dots\,.$$

Making use of the value of W given in (32), and remembering that

$$y_3a'-y_2b'=a,$$
$$y_4a'-y_2c'=2b,$$
$$y_5a'-y_2d'=3c, \text{ \&c.};$$

we see that

$$\left.\begin{aligned} x_2W'-y_2W=a^2\delta_c+\tfrac{1}{2}ab\,\delta_d+\tfrac{8}{5}(3ac+2b^2)\,\delta_e+\dots \\ +1.2a'a\delta_{c'}+(2.2a'b+1.3b'a)\delta_{d'}+(3.2a'c+2.3b'b+1.4c'a)\delta_{e'}+\dots \end{aligned}\right\}\dots(33).$$

This operator, W'' say, it is sometimes convenient to use, as in Arts. 23—26 below.

21. We can now form a few examples of semireciprocants of the (xz, y) class. Evidently a is one such. If

$$Aab'+Ba'b$$

is one, then, operating with W upon it,

$$2Aaa'^2+\tfrac{3}{2}Baa'^2=0,$$

therefore $A:B=3:-4$, and the function is

$$3ab'-4a'b.$$

Again, if $$Aac'+Bbb'+Cca'$$

is a semireciprocant, then, operating with W on it,

$$5Aaa'b'+B\,(2a'^2b+\tfrac{3}{2}aa'b')+\tfrac{1}{3}Ca'\,(8a'b+3ab')=0,$$

therefore $$2B+\tfrac{8}{3}C=0 \quad \text{and} \quad 5A+\tfrac{3}{2}B+C=0,$$

giving $A:B:C=3:-20:15$, so that the function required is

$$3ac'-20bb'+15ca'.$$

In a similar manner, it can be shown that

$$a'\,(4b^2-9ac)+6b'ab$$

is a semireciprocant, &c., &c.

We may also employ the method of eduction. If R is an absolute (xz, y) semireciprocant, expressed in terms of differential coefficients of y and z with respect to x, and if R' be its transform in terms of differential coefficients of y and x with respect to z, then

$$\frac{dR}{dx} = \frac{dz}{dx}\frac{dR'}{dz},$$

therefore

$$Z_1^{-\frac{1}{2}}\frac{dR}{dx} = x_1^{-\frac{1}{2}}\frac{dR'}{dz},$$

so that, if R' is an absolute (xz, y) semireciprocant, $x_1^{-\frac{1}{2}}\frac{dR'}{dz}$ is one too.

For example, a and a' are both semireciprocants, the first of index 6, the second of index 3; thus $\frac{a}{a'^2}$ is an absolute one. Therefore

$$\frac{a'^3}{2}\frac{d}{dz}\left(\frac{a}{a'^2}\right) = 4a'b - 3ab'$$

is a semireciprocant. Since its index is evidently 10 (for the total weights of $a, b, c, \ldots$ are 5, 6, 7, ... and their x-degrees are each 1, while the weights of $a', b', c', \ldots$ are 2, 3, 4, ... and their x-degrees each 1),

$$\frac{d}{dz}\left[\frac{(4a'b - 3ab')^3}{a'^{10}}\right]$$

will give another semireciprocant; this is, in fact,

$$a(9a'c' - 15b'^2) + 20ba'b' - 15ca'^2;$$

and by a similar process any number of such may be found. In applying the process, it is useful to remember that

$$\delta_z a' = 3b', \quad \delta_z b' = 4c', \quad \delta_z c' = 5d', \quad \delta_z d' = 5e', \text{ \&c.},$$

and that

$$\delta_z a = 2 \,.\, 4b,$$

$$2\delta_z b = 3 \,.\, 5c + \frac{3}{a'}(2b'b - c'a),$$

$$3\delta_z c = 4 \,.\, 6d + \frac{3}{a'}(3b'c - d'a),$$

$$4\delta_z d = 5 \,.\, 7e + \frac{3}{a'}(4b'd - e'a),$$

and so on.

In general, if R be an (xz, y) semireciprocant of index μ (*i.e.*, such that μ is equal to its total weight plus its x-degree),

$$\frac{d}{dz}(Ra^{-\mu 6})$$

will be another semireciprocant of the same kind; that is,

$$3a\frac{dR}{dz}-4\mu bR$$

will be one. Now, by the formulæ just given for differentiating $a, b, c \ldots, a', b', c' \ldots,$

$$\frac{dR}{dz}=\left(2.4b\delta_a+\frac{3.5}{2}c\delta_b+\frac{4.6}{3}d\delta_c+\ldots\right)R+(3b'\delta_{a'}+4c'\delta_{b'}+5d'\delta_{c'}+\ldots)R$$

$$+\frac{3}{a'}\left\{\left(b'b-\frac{c'a}{2}\right)\delta_b+\left(b'c-\frac{d'a}{3}\right)\delta_c+\left(b'd-\frac{e'a}{4}\right)\delta_d+\ldots\right\}R.$$

Also, if $\mu_a, \mu_b, \ldots \mu_{a'}, \mu_{b'}, \ldots$ be the indices of $a, b, \ldots a', b', \ldots,$

$$\mu R=(\mu_a\delta_a+\mu_b\delta_b+\ldots)R+(\mu_{a'}\delta_{a'}+\mu_{b'}\delta_{b'}+\ldots)R$$

$$=(6a\delta_a+7b\delta_b+8c\delta_c+\ldots)R+(3a'\delta_{a'}+4b'\delta_{b'}+5c'\delta_{c'}+\ldots)R,$$

therefore

$$3a\frac{dR}{dz}-4\mu bR=3(3ab'-4ba')\delta_{a'}+4(3ac'-4bb')\delta_{b'}+5(3ad'-4bc')\delta_{c'}+\ldots$$

$$+\tfrac{1}{2}(45ac-56b^2)\delta_b+\tfrac{1}{3}(72ad-96bc)\delta_c+\tfrac{1}{4}(105ae-144bd)\delta_d+\ldots$$

$$+9\frac{a}{a'}\left[\left(b'b-\frac{c'a}{2}\right)\delta_b+\left(b'c-\frac{d'a}{3}\right)\delta_c+\left(b'd-\frac{e'a}{4}\right)\delta_d+\ldots\right] \ldots.(34)$$

operating on R. The expression on the right-hand side of (34) is therefore a generator for pure (xz, y) semireciprocants.

22. I proceed now to the consideration of the full (x, y, z) reciprocants—*i.e.*, of the functions of $x_2, x_3, \ldots y_2, y_3, \ldots$ which persist when all three variables are simultaneously altered by substituting for each of them a linear function of $x, y,$ and z. These have been discussed by M. Halphen under the name of differential semi-invariants, on pp. 78 seqq. of the memoir cited at the beginning of the present paper. They will evidently not involve $x, y, x_1,$ or $y_1,$ and will be expressible in terms of $a, b, c, \ldots$ and $a', b', c' \ldots$. When so expressed they will be homogeneous and isobaric in both sets of letters, and will satisfy the

differential equations $Wf=0$, $W'f=0$, in addition to $Jf=0$. The system $Wf=0$, $W'f=0$ may conveniently be replaced by $Wf=0$, $W''f=0$, where W'' is the operator (33) found in Art. 20. It may be noticed that M. Halphen's notation may be connected with mine in the following way: his a_4, a_5, a_6, &c. are, in my notation, $\frac{2b}{a}$, $\frac{3c}{a}$, $\frac{4d}{a}$, &c.; and his b_4, b_5, b_6, &c. are $\frac{ac'-2bb'}{aa'}$, $\frac{ad'-3cb'}{aa'}$, $\frac{ae'-4db'}{aa'}$, &c.

23. To find the reciprocant which is linear in a, b, c, quadratic in a', b', c', and of extent 2 in each set of letters, assume it to be

$$a'(Aac+Bb^2)+Cb'ab+Dc'a^2.$$

Operating on this with W and W'', we find

$$Aa'a(\tfrac{8}{3}a'b+ab')+3Ba''ab+Ca(\tfrac{3}{2}aa'b'+2a''b)+5Da^2a'b'=0,$$

$$Aa^3a'+2Da^3a'=0,$$

so that $\quad \frac{8}{3}A+3B+2C=0,\ A+\frac{3}{2}C+5D=0,\ A+2D=0,$

whence $\quad A:B:C:D=18:-28:18:-9,$

and the reciprocant is

$$a'(18ac-28b^2)+18b'ab-9c'a^2.$$

This (except for the factor $6a^2a'$) is the same as M. Halphen's γ_3 or σ_3, given on p. 79 of his memoir.

24. To find the reciprocant linear in a, b, c and quadratic in a', b', c' and of extent 2 in each set of letters, assume it to be

$$a(Aa'c'+Bb'^2)+Cba'b'+Dca'^2.$$

Operating on it with W and W'', we find that

$$\tfrac{8}{3}D+2C=0,\quad 5A+4B+\tfrac{3}{2}C+D=0,\quad \text{and}\quad D+2A=0.$$

Therefore $\quad A:B:C:D=12:-21:32:-24,$

and the reciprocant is

$$a(12a'c'-21b'^2)+32ba'b'-24ca'^2.$$

It may be noticed that this is

$$2\tfrac{1}{5}a(4a'c'-5b'^2)-\tfrac{8}{5}a'(3ac'-20bb'+15ca');$$

and when written in this form it is seen at once to be annihilated by W; for the expression within the first bracket is an ordinary pure reciprocant, and that within the second bracket has been proved, in Art. 21, to be an (xz, y) semireciprocant.

25. To find the reciprocants of the form

$$a'(Aa^2d+Babc+Cb^3)+b'(Da^2c+Eab^2)+Fc'a^2b+Gd'a^3,$$

operate on this with W and W'', and we find

$$\tfrac{3}{2}B+2D+\tfrac{15}{4}A=0,\quad \tfrac{8}{3}B+\tfrac{9}{2}C+2E=0,\quad 2A+B+\tfrac{6}{3}D+3E+5F=0,$$

$$\tfrac{3}{4}A+\tfrac{3}{2}F+6G=0,\quad D+3G=0,$$

and

$$B+\tfrac{7}{2}A+2F+4G=0,\quad D+3G=0.$$

These give $A=-2F-8G$, $D=-3G$, $C=-\tfrac{8}{27}(7F+48G)$,

$$B=5F+24G,\quad E=-2F,$$

so that the constants are not absolutely determined, and there exists an infinite number of reciprocants of the form in question. Choosing $A=0$, we have

$$4a'b(27ac-40b^2)+27b'a(8b^2-3ac)-108c'a^2b+27d'a^3,$$

which is (except for the factor $27a^3a'$) the same as M. Halphen's δ_5 or τ_5, given on p. 79 of his memoir.

Choosing F and G so that $A:B:C=2:-9:8$, we have

$$a'(2a^2d-9abc+8b^3)+3b'a(ac-2b^2)+3c'a^2b-d'a^3 \quad \ldots\ldots(35),$$

which, except for the factor a^3a', is M. Halphen's ν (see p. 32 of his memoir).

26. Next, taking the form

$$a'\{Aa^3e+Ba^2bd+Ca^2c^2+Dab^2c+Eb^4\}+b'\{Fa^3d+Ga^2bc+Hab^3\}$$
$$+c'\{Ka^3c+La^2b^2\}+Md'a^3b+Ne'a^4,$$

and operating on it with W and W'' and equating the results to zero, we arrive at the following equations:

$$A=-\tfrac{5}{6}(3M+14N), \qquad F=-4N,$$
$$B=\tfrac{8}{3}(3M+16N), \qquad G=8N-3M,$$
$$C=6M+29N, \qquad H=\tfrac{16}{9}(3M+N),$$
$$D=-\tfrac{8}{9}(21M+124N), \qquad K=-4N,$$
$$E=\tfrac{16}{81}(33M+245N), \qquad L=-\tfrac{8}{9}(3M+N).$$

Taking $A = 0$, we have

$$A = 0, \quad D = 1872, \quad G = -1782, \quad L = -936,$$
$$B = -438, \quad E = -1456, \quad H = 1872, \quad M = 378,$$
$$C = -81, \quad F = 324, \quad K = 324, \quad N = -81.$$

This gives the reciprocant which is of extent 3 in a, b, c, ... and 4 in a', b', c', ..., and of the fourth and first degrees respectively in these two sets of letters.

If, on the other hand, we arrange so that $F : G : H = 2 : -9 : 8$, we may take

$$A = 10, \quad D = 144, \quad G = -54, \quad L = -24,$$
$$B = -48, \quad E = -80, \quad H = 48, \quad M = 10,$$
$$C = -27, \quad F = 12, \quad K = 12, \quad N = -3,$$

which are conveniently small numbers.

It may also be verified that

$$a'(a^3e - 4a^2bd - 3a^2c^2 + 12ab^2c - 6b^4) - 3a^2(c'a - b'b)^2$$
$$+ 4a'a(2ac - 3b^2)(c'a - b'b)$$
$$+ 3a'a^3(b'd - ae') + 6a^2ba'(ad' - b'c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

is annihilated by W and by W''. This (except for the factor $a^4a'^2$) is M. Halphen's s_7, given on p. 33 of his memoir.

27. The reciprocants (35) and (36) possess a further property, which is also possessed by the reciprocant a; they have been shown by M. Halphen to persist when the variables x, y, z are transformed by any homographic substitutions. They stand therefore in the same relation to the (x, y, z) reciprocants that principiants do to the ordinary or (x, y) reciprocants. Writing $\frac{x}{z}$, $\frac{y}{z}$, $\frac{1}{z}$ for x, y, z respectively, and proceeding in exactly the same manner as that employed by Professor Sylvester in the *American Journal of Mathematics*, Vol. IX., pp. 159, 160, we see that such a function F satisfies the equation $\Omega F = 0$, where

$$\Omega = x_2\frac{d}{dx_3} + 2x_3\frac{d}{dx_4} + 3x_4\frac{d}{dx_5} + \ldots + y_2\frac{d}{dy_3} + 2y_3\frac{d}{dy_4} + 3y_4\frac{d}{dy_5} + \ldots,$$

and that this condition is sufficient, as well as necessary, in order that a pure (x, y, z) reciprocant F should persist for homographic transformations of the variables. It can then be shown, as on p. 158 of the same *Journal*, that, in order that any function should be a Halphenian (x, y, z) reciprocant, it must be annihilated by J, W, W', and Ω; and that these four conditions are sufficient as well as necessary. There is no need to go through the whole of the work, as it is precisely similar to that given in the *Journal*. Since

$$\Omega a = 0, \quad \Omega b = a, \quad \Omega c = 2b, \quad \Omega d = 3c, \text{ \&c.},$$

it is seen that, when expressed in terms of $a, b, c, \ldots$ and $a', b', c', \ldots$, Ω takes the form

$$a\delta_b + 2b\delta_c + 3c\delta_d + \ldots + a'\delta_{b'} + 2b'\delta_{c'} + 3c'\delta_{d'} + \ldots \quad \ldots\ldots\ldots(37),$$

showing that every Halphenian reciprocant is a joint invariant of the two quantics

$$(a, b, c, d, \ldots)(x, y)^m \quad \text{and} \quad (a', b', c', d', \ldots)(x, y)^n.$$

Taking Ω in the form (37), it can readily be verified that both the reciprocants (35) and (36) are annihilated by Ω.

In fact, as Mr. Hammond has pointed out to me, (35) may be at once thrown into the form

$$5a'(a^2d - 3abc + 2b^3) - (a'\delta_a + b'\delta_b + c'\delta_c + d'\delta_d)\{a(a^2d - 3abc + 2b^3)\},$$

and I have found that (36) is equivalent to

$$\begin{aligned} &8a^2a'^2(ae - 4bd + 3c^2) - 3a^2(ac' + a'c - 2bb')^2 - 96a'^2(ac - b^2)^2 \\ &+ 48aa'(ac - b^2)(ac' + a'c - 2bb') \\ &- 3a'a^3(a'\delta_a + b'\delta_b + c'\delta_c + d'\delta_d + e'\delta_e)(ae - 4bd + 3c^2). \end{aligned}$$

I refrain from entering into further detail concerning this class of reciprocants, since they have been fully treated of by M. Halphen; and I have been careful to include in this article only such properties of them as are new.

Note on the Weierstrass Elliptic Functions, and their Applications.

By A. G. Greenhill.

[*Read March 10th*, 1887.]

The object of the present Note is to establish, from fundamental principles, the formulæ in the Weierstrass notation, corresponding to those given in Jacobi's notation by Mr. Glaisher in his "Note on the Functions Zu, Θu, $\Pi(u, a)$," read before this Society, Feb. 11th, 1886, and to show how naturally the formulæ arise from the definitions of the functions employed; afterwards, to apply the formulæ to some well known physical problems.

1. Starting with Euler's differential relation,

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

where
$$X=ax^4+bx^3+cx^2+dx+e,$$
$$Y=ay^4+by^3+cy^2+dy+e,$$

and with the integral relation obtained by Euler,

$$a(x+y)^2+b(x+y)+C=\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2,$$

which is the key-note of the Theory of Elliptic Functions; then, if X and Y are already of Weierstrass's canonical form, namely,

$$X=4x^3-g_2x-g_3,\quad Y=4y^3-g_2y-g_3,$$

this integral relation of Euler becomes, since now $a=0$, $b=4$,

$$4(x+y+z)=\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2,$$

writing $4z$ for C; this relation is symmetrical in x, y, and z; and may be written

$$x+y+z=\tfrac{1}{4}\left(\frac{\sqrt{Y}-\sqrt{Z}}{y-z}\right)^2$$
$$=\tfrac{1}{4}\left(\frac{\sqrt{Z}-\sqrt{X}}{z-x}\right)^2$$
$$=\tfrac{1}{4}\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2;$$

or
$$\begin{vmatrix} 1, & x, & \sqrt{X} \\ 1, & y, & \sqrt{Y} \\ 1, & z, & \sqrt{Z} \end{vmatrix} = 0;$$

leading to the symmetrical differential relation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} = 0;$$

so that, as shown by Professor Cayley, in *Crelle*, Vol. 87, p. 74, the integral of

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

may be considered as the particular case of

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} = 0;$$

where z is treated as an arbitrary constant, being the value of y corresponding to a certain value of x, in this case the infinite value.

2. If the general elliptic differential element $dx/\sqrt{U_x}$, where U_x denotes the general biquadratic expression in x, is not already of Weierstrass's canonical form, it may immediately be reduced to it by putting
$$s = -\frac{H_x}{U_x},$$

where H_x is the Hessian of the quartic U_x; and then (Cayley, *Elliptic Functions*, p. 346)

$$\frac{dx}{\sqrt{U_x}} = \tfrac{1}{2}\frac{ds}{\sqrt{(4s^3 - g_2 s - g_3)}},$$

where g_2 and g_3 are the quadrinvariant and the cubinvariant of the quartic X; also
$$4s^3 - g_2 s - g_3 = \frac{G_x^2}{U_x^3},$$

where G_x denotes the sextic covariant.

Euler's integral relation in the general case may therefore be put in the form,

$$\frac{H_x}{U_x} + \frac{H_y}{U_y} + \tfrac{1}{4}\left\{\frac{\dfrac{G_x}{U_x^{\frac{3}{2}}} - \dfrac{G_y}{U_y^{\frac{3}{2}}}}{\dfrac{H_x}{U_x} - \dfrac{H_y}{U_y}}\right\}^2 = \text{constant};$$

or symmetrically,

$$\begin{vmatrix} 1, & \frac{H_x}{U_x}, & \frac{G_x}{U_x^{\frac{3}{2}}} \\ 1, & \frac{H_y}{U_y}, & \frac{G_y}{U_y^{\frac{3}{2}}} \\ 1, & \frac{H_z}{U_z}, & \frac{G_z}{U_z^{\frac{3}{2}}} \end{vmatrix} = 0.$$

3. Introducing now the notation of Weierstrass, where if

$$u = \int_x^\infty \frac{dx}{\sqrt{(4x^3 - g_2x - g_3)}},$$

the canonical elliptic integral of the first kind, then x is an elliptic function of u which is denoted by $\wp u$; so that

$$x = \wp u,$$

and

$$\frac{dx}{du} = \wp' u$$

$$= -\sqrt{(4x^3 - g_2x - g_3)} = -\sqrt{X}.$$

Then, if $y = \wp v$, $z = \wp w$, we have

$$u + v + w = 0,$$

and Euler's integral relation becomes

$$\wp u + \wp v + \wp w = \tfrac{1}{4}\left(\frac{\wp' v - \wp' w}{\wp v - \wp w}\right)^2$$

$$= \tfrac{1}{4}\left(\frac{\wp' w - \wp' u}{\wp w - \wp u}\right)^2$$

$$= \tfrac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2;$$

or,

$$\begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v, & \wp' v \\ 1, & \wp w, & \wp' w \end{vmatrix} = 0.$$

4. Since $\wp u$ is an even function of u, therefore

$$\wp w = \wp (u + v),$$

so that

$$\wp (u+v) + \wp u + \wp v = \tfrac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2;$$

and, changing the sign of v, since $\wp' v$ is an odd function of v, therefore

$$\wp(u-v)+\wp u+\wp v = \tfrac{1}{4}\left(\frac{\wp' u+\wp' v}{\wp u-\wp v}\right)^2.$$

Subtracting these equations, we obtain

$$\wp(u-v)-\wp(u+v) = \frac{\wp' u\,\wp' v}{(\wp u-\wp v)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A),$$

the equation analogous to Mr. Glaisher's equation (A) (p. 153), which he takes as his starting point.

5. Integrating our equation (A) with respect to v,

$$\zeta(u+v)+\zeta(u-v)+C = \frac{\wp' u}{\wp u-\wp v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A_1),$$

where C is the constant of integration, independent of v, and the function ζu is defined by

$$\zeta u = -\int \wp u\, du, \qquad \zeta' u = -\wp u,$$

(Halphen, *Traité des Fonctions Elliptiques et de leurs Applications*, p. 135, Paris, 1886); so that ζu is analogous to Jacobi's function Zu.

To determine the constant C in (A_1), put $v=0$; then $\wp v = \infty$, and $C = -2\zeta u$, so that

$$\zeta(u+v)+\zeta(u-v)-2\zeta u = \frac{\wp' u}{\wp u-\wp v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

analogous to Mr. Glaisher's equation (β).

6. Putting $v=u$ in (A_1), then C assumes the indeterminate form $\infty-\infty$, and must be evaluated; we shall find, eventually,

$$\zeta(u+v)+\zeta(u-v)-\zeta 2u = -\tfrac{1}{2}\frac{\wp'(u+v)-\wp'(u-v)}{\wp(u+v)-\wp(u-v)} \quad \ldots\ldots (\alpha),$$

or

$$= \frac{-\wp' u}{\wp u-\wp v} + \tfrac{1}{2}\frac{\wp'' u}{\wp' u},$$

analogous to Mr. Glaisher's equation (α). For, since

$$\zeta(u+v)+\zeta(u-v)-2\zeta u = \frac{\wp' u}{\wp u-\wp v},$$

therefore, interchanging u and v,

$$\zeta(u+v)-\zeta(u-v)-2\zeta v = \frac{-\wp' v}{\wp u-\wp v};$$

and therefore, by addition,

$$\zeta(u+v)-\zeta u-\zeta v = \tfrac{1}{2}\frac{\wp' u-\wp' v}{\wp u-\wp v},$$

or

$$= \surd\{\wp(u+v)+\wp u+\wp v\};$$

and, changing u and v into $u+v$ and $u-v$,

$$\zeta 2u-\zeta(u+v)-\zeta(u-v) = \tfrac{1}{2}\frac{\wp'(u+v)-\wp'(u-v)}{\wp(u+v)-\wp(u-v)}$$

$$= \tfrac{1}{2}\frac{\wp'' u}{\wp' u} - \frac{\wp' u}{\wp u-\wp v}.$$

7. Next, introducing Weierstrass's function σu, defined by

$$\log \sigma u = \int_0 \zeta u\, du,$$

or

$$\sigma u = \exp \int_0 \zeta u\, du,$$

so that Weierstrass's function σu is analogous to Jacobi's function Θu, or rather Hu; then

$$\zeta u = \frac{\sigma' u}{\sigma u} = \frac{d}{du}\log \sigma u,$$

and

$$\wp u = -\zeta' u = -\frac{d^2}{du^2}\log \sigma u.$$

Integrating equation (β) with respect to u between the limits 0 and u, we find

$$\log\frac{\sigma(v+u)}{\sigma v} + \log\frac{\sigma(v-u)}{\sigma v} - 2\log \sigma u = \log(\wp u-\wp v),$$

so that

$$\wp u-\wp v = \frac{\sigma(v+u)\,\sigma(v-u)}{\sigma^2 v\,\sigma^2 u},$$

or

$$= -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (\beta_1),$$

the fundamental formula in the use of Weierstrass's elliptic functions, analogous to Mr. Glaisher's equation (β_1).

8. By integrating (β) with respect to v instead of with respect to u, we find

$$\int \zeta(u+v)\,dv + \int \zeta(u-v)\,dv - 2v\zeta u = \int \frac{\wp' u\,dv}{\wp u-\wp v},$$

or $$\int_0 \frac{\wp' u\, dv}{\wp u - \wp v} = \log \frac{\sigma(u+v)}{\sigma(u-v)} - 2v\zeta u \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta_2),$$

the elliptic integral of the third kind in the Weierstrass notation, corresponding to Jacobi's $\Pi(v, u)$.

By integrating the equation in § 6,

$$\tfrac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v} = \zeta(u+v) - \zeta u - \zeta v,$$

with respect to u and v respectively, we obtain

$$\int_0 \tfrac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v}\, du = \log \frac{\sigma(u+v)}{\sigma u} - u\zeta v,$$

$$\int_0 \tfrac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v}\, dv = \log \frac{\sigma(u+v)}{\sigma v} - v\zeta u,$$

either of which may be taken as Weierstrass's canonical form of the elliptic integral of the third kind; and these two integrals illustrate the theorem analogous to that obtained by the interchange of amplitude u and parameter v in Jacobi's elliptic integral of the third kind $\Pi(u, v)$.

Or, interchanging u and v in (β_2),

$$\int \frac{\wp' v\, du}{\wp u - \wp v} = \log \frac{\sigma(u-v)}{\sigma(u+v)} + 2u\zeta v;$$

so that, by addition,

$$\int \frac{\wp' u\, dv + \wp' v\, du}{\wp u - \wp v} = 2u\zeta v - 2v\zeta u,$$

corresponding to interchange of *amplitude* and *parameter*.

9. Supposing $X = 4x^3 - g_2 x - g_3$ resolved into three real linear factors $4(x-e_1)(x-e_2)(x-e_3)$, where $e_1 > e_2 > e_3$, and denoting by ω_1 and ω_3 the real and imaginary half-periods of the elliptic functions, then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dx}{\sqrt{X}}, \quad \omega_1 + \omega_3 = \int_{e_2}^{\infty} \frac{dx}{\sqrt{X}}, \quad 2\omega_1 + \omega_3 = \int_{e_3}^{\infty} \frac{dx}{\sqrt{X}},$$

$$2\omega_1 + 2\omega_3 = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{X}};$$

so that
$$\omega_1 = \int_{e_1}^{\infty} \frac{dx}{\sqrt{X}} = \int_{e_2}^{e_3} \frac{dx}{\sqrt{X}},$$
$$\omega_3 = \int_{e_2}^{e_1} \frac{dx}{\sqrt{X}} = \int_{-\infty}^{e_3} \frac{dx}{\sqrt{X}};$$

and, consequently,
$$\wp\omega_1 = e_1, \quad \wp(\omega_1+\omega_3) = e_2, \quad \wp\omega_3 = e_3;$$

while
$$\wp'\omega_1 = \wp'(\omega_1+\omega_3) = \wp'\omega_3 = 0.$$

10. Returning to the fundamental formula
$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v},$$

we may suppose the right-hand side resolved into the factors
$$\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(-u\zeta v) \quad \text{and} \quad \frac{\sigma(u-v)}{\sigma u\,\sigma v}\exp(u\zeta v);$$

so that, denoting $\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(-u\zeta v)$ by ϕu or $\phi(u, v)$, then
$$\phi(u, -v) = \phi(-u, v) = -\frac{\sigma(u-v)}{\sigma u\,\sigma v}\exp(u\zeta v),$$

and
$$\wp u - \wp v = \phi(u, v)\,\phi(u, -v).$$

Also, replacing v by the particular values ω_1, $\omega_1+\omega_3$, and ω_3, we obtain
$$\phi u = \frac{\sigma_1 u}{\sigma u}, \quad \frac{\sigma_2 u}{\sigma u}, \quad \frac{\sigma_3 u}{\sigma u},$$

or
$$\phi u = \surd(\wp u - e_1), \quad \surd(\wp u - e_2), \quad \surd(\wp u - e_3);$$

and then ϕu is a doubly periodic function.

But, for any other value of v, the function $\phi(u, v)$ is multiplied by a constant factor when the variable is increased by a complete period $2\omega_1$ or $2\omega_3$ (Halphen, p. 227); this function was introduced by Hermite, and called by him a doubly periodic function of the second kind (*fonction doublement périodique de second espace*).

We see, from § 8, that its logarithm is an elliptic integral of the third kind; for
$$\log\phi(u, v) = \log\frac{\sigma(u+v)}{\sigma u\,\sigma v} - u\zeta v$$
$$= \int \tfrac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v}\,du.$$

11. This function φu is shown by Halphen (*Fonctions elliptiques*, p. 235) to satisfy the equation

$$\frac{d^2\phi}{du^2} = (2\wp u + \wp v)\,\phi,$$

Lamé's differential equation for $n = 1$, with Weierstrass's notation.

Lamé's general differential equation with Jacobi's elliptic function notation

$$\frac{d^2y}{dx^2} = \{n\,(n+1)\,k^2\,\mathrm{sn}^2\,x + h\}\,y$$

is considered by Hermite in *Sur quelques applications des fonctions elliptiques*, Paris, 1885, *Premier fascicule*. This is a reprint of Hermite's papers which appeared in the *Comptes Rendus*, beginning in 1877.

By the use of Weierstrass's notation, the work of Hermite is considerably simplified. The equation of Lamé then becomes

$$\frac{d^2y}{dx^2} = \{n\,(n+1)\,\wp x + h\}\,y,$$

and, in accordance with Hermite's results, the solution of this equation is

$$y = CF(x) + C'F(-x),$$

where C and C' are arbitrary constants, and

$$F(x) = \left(\frac{d}{dx}\right)^{n-1}\Phi x - A_1\left(\frac{d}{dx}\right)^{n-3}\Phi x + A_2\left(\frac{d}{dx}\right)^{n-5}\Phi x \ldots,$$

where

$$\Phi x = \frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega}\exp(\lambda - \zeta\omega)\,x = e^{\lambda x}\phi\,(x, \omega);$$

also $A_1, A_2, \ldots, \lambda$, and $\wp\omega$ are certain definite constants, rational functions of h and the invariants g_2 and g_3.

12. A great part of Hermite's work is devoted to the determination of these constants for the successive integral values of n, the complexity of the work increasing very rapidly.

For $n = 1$ we have just seen that the solution of the corresponding equation of Lamé,

$$\frac{d^2y}{dx^2} = (2\wp x + h)\,y,$$

is obtained, putting $h = \wp v$, in the form

$$y = C\phi(x, v) + C'\phi(-x, v),$$

where $$\phi(x, v) = \frac{\sigma(x+v)}{\sigma x \sigma v} \exp(-x\zeta v), \text{ as above.}$$

For $n = 2$, the equation becomes

$$\frac{d^2y}{dx^2} = (6\wp x + h)\, y;$$

and, as we have seen in these *Proceedings*, June 10th, 1886, p. 374, the solution is then, putting $h = -3\wp(a-b)$,

$$y = C \frac{\sigma(x+a)\,\sigma(x+b)}{\sigma a\, \sigma b\, \sigma^2 x} \exp(-\zeta a - \zeta b)\, x$$

$$+ C' \frac{\sigma(x-a)\,\sigma(x-b)}{\sigma a\, \sigma b\, \sigma^2 x} \exp(\ \ \zeta a + \zeta b)\, x,$$

subject to the condition that

$$\wp' a = -\wp' b, \text{ and therefore} = \wp'(a-b);$$

and this, by Halphen's equation (36) (*Fonctions elliptiques*, p. 230), can be thrown into the form

$$y = C \frac{d}{dx}\left\{\frac{\sigma(x+\omega)}{\sigma x\, \sigma\omega} \exp(-\zeta a - \zeta b)\, x\right\}$$

$$+ C' \frac{d}{dx}\left\{\frac{\sigma(x-\omega)}{\sigma x\, \sigma\omega} \exp(\ \ \zeta a - \zeta b)\, x\right\},$$

where $$\omega = a+b,$$

agreeing thus with Hermite's form of the result, if

$$\lambda = \zeta\omega - \zeta a - \zeta b = \tfrac{1}{2}\frac{\wp' a - \wp' b}{\wp a - \wp b} = \frac{\wp' a}{\wp a - \wp b};$$

so that $$\zeta(a-b) = \ \ \zeta a - \zeta b,$$

also $$\wp(a-b) = -\wp a - \wp b.$$

13. It is interesting to verify that the above value of y is a solution of the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = 6\wp x - 3\wp(a-b).$$

Taking the particular solution

$$y = \frac{\sigma(x+a)\,\sigma(x+b)}{\sigma a\,\sigma b\,\sigma^2 x}\exp\left(-\zeta a-\zeta b\right)x,$$

then
$$\frac{1}{y}\frac{dy}{dx} = \zeta(x+a)-\zeta x-\zeta a+\zeta(x+b)-\zeta x-\zeta b$$
$$= \tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}+\tfrac{1}{2}\frac{\wp' x-\wp' b}{\wp x-\wp b};$$

and, differentiating again,

$$\frac{1}{y}\frac{d^2y}{dx^2} = \left(\frac{1}{y}\frac{dy}{dx}\right)^2+2\wp x-\wp(x+a)-\wp(x+b)$$
$$= \left(\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}+\tfrac{1}{2}\frac{\wp' x-\wp' b}{\wp x-\wp b}\right)^2+2\wp x-\wp(x+a)-\wp(x+b)$$
$$= 4\wp x+\wp a+\wp b+\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}\frac{\wp' x-\wp' b}{\wp x-\wp b}.$$

But, if
$$\wp' a = -\wp' b = \wp'(a-b),$$
$$\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a}\frac{\wp' x-\wp' b}{\wp x-\wp b} = \tfrac{1}{2}\frac{\wp'^2 x-\wp'^2 a}{(\wp x-\wp a)(\wp x-\wp b)}$$
$$= 2(\wp x+\wp a+\wp b),$$

in consequence of the relations

$$\wp'^2 a = \wp'^2 b,$$

or
$$4\wp^3 a-g_2\wp a-g_3 = 4\wp^3 b-g_2\wp b-g_3;$$

or
$$4(\wp^3 a+\wp a\,\wp b+\wp^2 b) = g_2,$$

so that
$$\wp'^2 x-\wp'^2 a = (\wp x+\wp a)\left\{4(\wp^2 x+\wp a\,\wp x+\wp^2 a)-g_2\right\}$$
$$= 4(\wp x-\wp a)(\wp^3 x+\wp a\,\wp x-\wp a\,\wp b-\wp^2 b)$$
$$= 4(\wp x-\wp a)(\wp x-\wp b)(\wp x+\wp a+\wp b).$$

Therefore
$$\frac{1}{y}\frac{d^2y}{dx^2} = 6\wp x+3\wp a+3\wp b$$
$$= 6\wp x-3\wp(a-b),$$

since
$$\wp(a-b)+\wp a+\wp b = 0,$$

in consequence of the relation

$$\wp' a = -\wp' b = \wp'(a-b).$$

When $\wp'(a-b)=0$, the solution of the differential equation is an algebraical function of $\wp x$, and we obtain the particular solutions considered by Lamé.

14. If, however, we turn to p. 106 of Hermite's work, we must replace h by $6\wp a$, in order to obtain an analogous solution; and then Lamé's differential equation, in the form

$$\frac{1}{y}\frac{d^2y}{dx^2}=6\wp x+6\wp a,$$

has the solution

$$\begin{aligned} y= &\quad C\frac{d}{dx}\left\{\frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega}\exp\left(\quad\lambda-\zeta\omega\right)x\right\}\\ &+C'\frac{d}{dx}\left\{\frac{\sigma(x-\omega)}{\sigma x\,\sigma\omega}\exp\left(-\lambda+\zeta\omega\right)x\right\}\\ &=\quad C\frac{d}{dx}\left\{\phi(x,\omega)\,e^{\lambda x}\right\}+C'\frac{d}{dx}\left\{\phi(-x,\omega)\,e^{-\lambda x}\right\}; \end{aligned}$$

subject to the conditions

$$\lambda^2=\tfrac{1}{4}\frac{\wp''^2\omega}{(\wp\omega-\wp a)^2}=\wp\omega+2\wp a,$$

so that $$\lambda=-\tfrac{1}{2}\frac{\wp'\omega}{\wp\omega-\wp a}=-\tfrac{1}{2}\zeta(\omega+a)-\tfrac{1}{2}\zeta(\omega-a)+\zeta\omega,$$

or $$\lambda-\zeta\omega=-\tfrac{1}{2}\zeta(\omega+a)-\tfrac{1}{2}\zeta(\omega-a);$$

also $$\wp\omega=\frac{4\wp^3a+\frac{1}{2}g_3}{\wp''a},$$

or $$\wp\omega-\wp a=-\tfrac{1}{2}\frac{\wp'^2a}{\wp''a},$$

giving $\wp\omega$, and consequently λ, in terms of $\wp a$ or h.

15. For $n=3$, the identification with Hermite's results (*Sur quelques applications, &c.*, pages 120 to 129) is obtained by writing Lamé's corresponding equation

$$\frac{1}{y}\frac{d^2y}{dx^2}=12\wp x+h,$$

and then $$h=-5l=15\wp a,$$

or $$l=-3\wp a,$$

so that the differential equation is

$$\frac{1}{y}\frac{d^2y}{dx^2} = 12\,\wp x + 15\,\wp a.$$

This suggests, by analogy, that the generalised form of Lamé's should be written, following Brioschi, in the form

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\,\wp x + n(2n-1)\,\wp a,$$

agreeing with the above forms for $n = 1, 2, 3$.

Then (Hermite, p. 120),

$$s_0 = -e_3, \quad s_1 = \frac{g_2}{20}, \quad s_2 = \frac{g_3}{28}, \quad s_3 = \frac{g_2^2}{600}, \ \ldots;$$

and (Hermite, p. 124)

$$\Omega = \wp\omega, \quad \Omega_1 = \tfrac{1}{2}\wp'\omega, \quad \Omega_2 = \wp^2\omega - \tfrac{1}{5}g_2, \quad \Omega_3 = \tfrac{1}{2}\wp\omega\,\wp'\omega, \ \ldots,$$

the same as the coefficients given by Halphen (p. 231) in the expansion in ascending powers of u of $\phi(u; \omega)$.

Then, when $n = 3$ (Hermite, p. 126), the identification of result is made by means of the equations

$$\lambda^2 - 3\,\wp\omega - \frac{\wp'\omega}{\lambda} = -3l = 9\,\wp a = \tfrac{3}{5}h,$$

$$h_1 = -\tfrac{1}{5}h, \quad a = \tfrac{3}{4}g_2, \quad b = \tfrac{27}{4}g_3, \quad 4a^3 - b^2 = \tfrac{27}{16}\Delta,$$

$$D = l^2 - a = \tfrac{3}{2}\wp''a, \quad S = -108\,\wp a, \ \ldots,$$

$$\wp\omega = -\frac{\psi(l)}{36b\,(b^2-a)^2}, \quad \wp\omega - e_3 = -\frac{PA^2}{SD^2}, \quad \wp\omega - e_2 = \frac{QB^2}{SD^2},$$

$$\wp\omega - e_1 = \frac{RC^2}{SD^2}, \quad \wp'\omega = -\frac{2ABC\lambda}{SD^2}, \ \ldots.$$

The solution of the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = 12\,\wp x + h$$

is, therefore,

$$y = CF(x) + C'F(-x),$$

where

$$F(x) = \frac{d^2}{dx^2}\Phi x - \tfrac{1}{5}h\Phi x,$$

and
$$\Phi x = \frac{\sigma(x+\omega)}{\sigma x\, \sigma \omega} \exp(\lambda - \zeta\omega)\, x$$
$$= \phi(x, \omega) \exp \lambda x,$$

where
$$h = \tfrac{5}{3}\left(\lambda^2 - 3\wp\omega - \frac{\wp'\omega}{\lambda}\right), \ \ldots .$$

The direct verification that Fx is a particular solution of this differential equation is an interesting piece of analysis, but it is omitted here, as the work is rather long.

16. For higher integral values of n, the determination of $\wp\omega$ and λ in the solution of Lamé's differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\wp x + h,$$

in the form
$$y = CF(x) + C'F(-x),$$

and
$$F(x) = \left(\frac{d}{dx}\right)^{n-1} \Phi x - A_1 \left(\frac{d}{dx}\right)^{n-3} \Phi x + \ldots ,$$

where
$$\Phi x = \frac{\sigma(x+\omega)}{\sigma x\, \sigma \omega} \exp(\lambda - \zeta\omega)\, x,$$

increases rapidly in complication, and the general solution has not yet been obtained by Hermite; although he attempts the solution from the consideration of the product $F(x)\, F(-x)$ of two particular solutions, this product being in all cases a doubly periodic function.

17. The physical origin of Lamé's differential equation is explained in Maxwell's *Electricity and Magnetism*, Todhunter's *Functions of Laplace, Lamé, and Bessel*, Ferrers's *Spherical Harmonics*, and Heine's *Kugelfunctionen*.

Transforming Poisson's equation to ellipsoidal coordinates (Maxwell, I., p. 181), and employing Weierstrass's notation as explained in these *Proceedings*, June 10, 1886 (Vol. XVII., p. 378), the equation becomes

$$(\mu - \nu)\frac{d^2\phi}{du^2} + (\nu - \lambda)\frac{d^2\phi}{dv^2} + (\lambda - \mu)\frac{d^2\phi}{dw^2} = 0,$$

(*cf.* Klein, *Ueber Lamé'sche Functionen*, *Math. Ann.*, XVIII.).

Here λ, μ, ν are the three roots of the cubic equation

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1,$$

defining the three confocal quadric surfaces through a point; and then, as has been shown, we should put

$$a^2+\lambda = \wp u - e_1, \quad b^2+\lambda = \wp u - e_2, \quad c^2+\lambda = \wp u - e_3,$$

$$a^2+\mu = \wp v - e_1, \quad b^2+\mu = \wp v - e_2, \quad c^2+\mu = \wp v - e_3,$$

$$a^2+\nu = \wp w - e_1, \quad b^2+\nu = \wp w - e_2, \quad c^2+\nu = \wp w - e_3;$$

and, in going round three sides of the boundary of the *period rectangle*,

$u = r\omega_1$, for the confocal ellipsoids,

$v = \omega_1 + s\omega_3$, for the hyperboloids of one sheet,

$w = t\omega_1 + \omega_3$, for the hyperboloids of two sheets;

r, s, t denoting real proper fractions, the fourth side of the period rectangle giving imaginary surfaces; so that at two corners of the period rectangle $r = 1$ or $s = 0$ gives the focal ellipse, $s = 1$ or $t = 1$ the focal hyperbola.

Then Poisson's equation becomes of the symmetrical form

$$(\wp v - \wp w)\frac{d^2\phi}{du^2} + (\wp w - \wp u)\frac{d^2\phi}{dv^2} + (\wp u - \wp v)\frac{d^2\phi}{dw^2} = 0.$$

18. Lamé supposes that ϕ is the product of three functions U, V, W, such that U is a function of u only, V of v only, and W of w only; and then the equation becomes

$$(\wp v - \wp w)\frac{d^2U}{U\,du^2} + (\wp w - \wp u)\frac{d^2V}{V\,dv^2} + (\wp u - \wp v)\frac{d^2W}{W\,dw^2} = 0,$$

equivalent to

$$\frac{d^2U}{U\,du^2} = g\,\wp u + h,$$

$$\frac{d^2V}{V\,dv^2} = g\,\wp v + h,$$

$$\frac{d^2W}{W\,dw^2} = g\,\wp w + h,$$

where g and h are arbitrary constants; and g is replaced by $n\,(n+1)$, where n is an integer, by analogy with Laplace's equation, in order that the integral should be a *uniform* function; and we thus obtain three of Lamé's differential equations of the same form.

19. The application of Lamé's equation for $n = 2$ to the dynamical problem of the motion of the spherical pendulum is given by Hermite (p. 109), and adapted to Weierstrass's notation in these *Proceedings*, June 10, 1886 (Vol. XVII., p. 374).

The equation of the projection of the motion of the bob of the pendulum on a horizontal plane is there given in the form

$$x+iy = 2il\,\frac{\sigma(u+a)\,\sigma(u+b)}{\sigma a\,\sigma b\,\sigma^2 u}\exp(-\zeta a-\zeta b)\,u,$$

where, 2τ denoting a complete period of the motion, we must put

$$u = t\omega_1/\tau+\omega_3;$$

also

$$a = \omega_1+r\omega_3, \quad b = s\omega_3,$$

r and s denoting real proper fractions.

In the spherical pendulum the additional condition $\wp' a = -\wp' b$ is required, but in the representation of the general motion of a point on the axis of a top, projected on a horizontal plane, this restriction is not required, while the form of the equation remains unaltered; but it is no longer the solution of Lamé's equation, unless referred to moving axes rotating with a certain constant angular velocity about the vertical.

20. It is unfortunate, at first sight, that the parameters like a and b required in the solution of these and similar dynamical problems are always imaginary; but this inconvenience disappears when we employ the above form for $x+iy$ as a function of t.

As dynamical illustrations of Lamé's equation for $n = 1$, Hermite considers (i.) the problem of the motion of a body under no forces, including the equation of Poinsot's herpolhode; (ii.) the equation of the tortuous Elastica in equilibrium under balancing forces and couples at its ends.

21. Let us consider, first, the equations of the Elastica (Hermite, *Sur quelques applications, &c.*, p. 93). These equations may be written

$$y'z''-y''z' = \alpha x'+\beta y,$$
$$z'x''-z''x' = \alpha y'-\beta x,$$
$$x'y''-x''y' = \alpha z'+\gamma,$$

where the accents denote differentiation with respect to the arc s, and α, β, γ denote constants depending on the flexibility of the wire, and the impressed forces; also the axis of z is taken along the central axis of the applied wrench at any point of the curve.

These equations are due in this form to Binet and Wantzel (*Comptes Rendus*, 1844), and may be established as follows: denoting by B the flexural rigidity of the wire, by Z and N the impressed force and couple, and by G the tangential torsional couple of the wire, then the

equations of equilibrium are, taking component moments at any point xyz of the wire,

$$B(y'z''-y''z') = Gx'+Zy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$B(z'x''-z''x') = Gy'-Zx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$B(x'y''-x''y') = Gz'+N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

equations of the above form.

Differentiating with respect to s, multiplying respectively by x', y', z', and adding, gives $G'=0$; so that G is constant.

Multiplying (1) by x', (2) by y', and (3) by z', and adding, gives

$$G-Z(xy'-x'y) = 0,$$

so that $$xy'-x'y = r^2\frac{d\theta}{ds} = h, \text{ a constant};$$

and therefore $$xy''-x''y = 0.$$

Again, multiplying (1) by x, and (2) by y, and adding, gives

$$Bz''(xy'-x'y)-Bz'(xy''-x''y) = G(xx'+yy'),$$

or $$Bz'' = Z(xx'+yy'),$$

so that $$Bz' = \tfrac{1}{2}Z(x^2+y^2)+H.$$

Then
$$B^2z'^2 = Z^2\{(x^2+y^2)(x'^2+y'^2)-(xy'-x'y)^2\}$$
$$= 2Z(Bz'-H)(1-z'^2)-Z^2h^2,$$

so that z' is an elliptic function of the arc s.

We may, if we like, modify Hermite's formulæ to the Weierstrass notation, and obtain from these equations

$$x+iy = C\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\lambda-\zeta\omega)u,$$

$$z = 2\zeta u+\gamma u,$$

where $$u = s\omega_1/c+\omega_3,$$

$2c$ denoting the length of a complete wave of the elastica; or we may obtain these equations immediately from the previous solution for the top (§ 19), by means of Kirchhoff's *Kinetic Analogue*, which asserts that the tangent of the elastica, if properly orientated, can be made to keep always parallel to the axis of the top, provided that the point of contact on the elastica moves with a certain definite constant velocity, and provided that certain initial conditions are satisfied.

Then, in the Elastica, in consequence of the Kinetic Analogue,

$$\frac{d}{du}(x+iy) = C\,\frac{d}{du}\,\frac{\sigma(u+a+b)}{\sigma u\,\sigma(a+b)}\,\exp(-\zeta a-\zeta b)\,u,$$

$$\frac{dz}{du} = -2\wp u+\gamma,$$

(*Proc. Lond. Math. Soc.*, June 10th, 1886); and, integrating,

$$x+iy = C\,\frac{\sigma(u+a+b)}{\sigma u\,\sigma(a+b)}\,\exp(-\zeta a-\zeta b)\,u,$$

$$z = 2\zeta u+\gamma u,$$

where du/ds is constant, and therefore

$$u = s\omega_1/c+\omega_3;$$

the imaginary constant ω_3 being added in order that dz/ds should oscillate between finite limits.

22. In the motion of a rigid body about a fixed point under no forces, the solution of Euler's equations of motion,

$$A\frac{dp}{dt}-(B-C)\,qr = 0, \quad B\frac{dq}{dt}-(C-A)\,rp = 0,$$

$$C\frac{dr}{dt}-(A-B)\,pq = 0,$$

has already been given in these *Proceedings*, June 10th, 1886 (Vol. XVII., p. 366), in the form

$$Ap^2 = -m^2(B-C)(\wp u-e_1),$$

$$Bq^2 = -m^2(C-A)(\wp u-e_2),$$

$$Cr^2 = -m^2(A-B)(\wp u-e_3),$$

where the factor m^2 is now introduced for homogeneity; and then

$$\frac{du^2}{dt^2} = -m^2\,\frac{(B-C)(C-A)(A-B)}{ABC} = \mu^2,$$

suppose; so that

$$u = \mu t+\omega_3$$

$$= t\omega_1/\tau+\omega_3,$$

2τ denoting the complete period of the motion, and the imaginary constant ω_3 being added in order to make $\wp u$ oscillate between e_2 and e_3, and the polhode consequently enclose the principal axis A.

Also, e_1, e_2, e_3, g_2, and g_3 are determined, as before, from the relations

$$Ap^2+Bq^2+Cr^2=T, \quad A^2p^2+B^2q^2+C^2r^2=G^2,$$

where $\frac{1}{2}T$ and G denote the constant kinetic energy and resultant angular momentum of the system.

23. For the herpolhode, take the equation of § 532 (Routh's *Rigid Dynamics*, 3rd edition),

$$\frac{d\phi}{dt}=\frac{T}{G}+\frac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCGT^2}\cot^2\zeta,$$

and we shall find (§ 25) that this can easily be thrown into the form

$$\frac{di\phi}{du}=\lambda-\tfrac{1}{2}\frac{\wp'\omega}{\wp u-\wp\omega}$$
$$=\lambda-\zeta\omega+\tfrac{1}{2}\zeta(u+\omega)-\tfrac{1}{2}\zeta(u-\omega),$$

so that
$$i\phi=(\lambda-\zeta\omega)\,u+\tfrac{1}{2}\log\frac{\sigma(u+\omega)}{\sigma(u-\omega)},$$

or
$$e^{i\phi}=\sqrt{\left\{\frac{\sigma(u+\omega)}{\sigma(u-\omega)}\right\}}\exp(\lambda-\zeta\omega)\,u.$$

Then
$$\lambda^2=-\frac{T^2}{G^2}\frac{dt^2}{du^2}=\frac{ABCT^2}{m^2G^2(B-C)(C-A)(A-B)},$$

and ω is the (imaginary) value of u which makes x^2+y^2 in the herpolhode vanish. So that

$$x^2+y^2=k^2(\wp\omega-\wp u)=k^2\frac{\sigma(u+\omega)\,\sigma(u-\omega)}{\sigma^2u\,\sigma^2\omega},$$

and therefore
$$x+iy=k\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\lambda-\zeta\omega)\,u,$$

or
$$x-iy=k\frac{\sigma(u-\omega)}{\sigma u\,\sigma\omega}\exp(-\lambda+\zeta\omega)\,u,$$

where
$$u=t\omega_1/\tau+\omega_3,$$

is the equation of the herpolhode.

24. Referred to axes rotating with constant angular velocity T/G, the equation of the herpolhode will be

$$x+iy=k\phi(u,\omega),$$

or
$$x-iy=k\phi(-u,\omega),$$

so that now x and y satisfy Lamé's equation for $n = 1$,

$$\frac{1}{x}\frac{d^2x}{du^2} = \frac{1}{y}\frac{d^2y}{du^2} = 2\wp u + \wp\omega,$$

and this relative herpolhode will be described therefore exactly as by a particle properly projected under a central attraction proportional to

$$r(2\wp u + \wp\omega) = 3r\wp\omega - 2r^3/k^2$$

(compare an article by Pinczon in the *Comptes Rendus*, 12th April, 1887, on the "Herpolhode").

25. In order to prove the equation

$$\frac{di\phi}{du} = \lambda - \tfrac{1}{2}\frac{\wp'\omega}{\wp u - \wp\omega},$$

we notice that we must put

$$\lambda = -i\frac{T}{G}\frac{dt}{du} = -i\frac{T}{Gm}\sqrt{\left\{\frac{ABC}{-(B-C)(C-A)(A-B)}\right\}};$$

also, in the herpolhode (Routh, p. 409),

$$\begin{aligned} x^2+y^2 = r^2-p^2 &= \frac{M\epsilon^4}{T}\left(\omega^2 - \frac{T^2}{G^2}\right) \\ &= -\frac{M\epsilon^4}{T}\left\{m^2\frac{B-C}{A}(\wp u - e_1) + m^2\frac{C-A}{B}(\wp u - e_2)\right. \\ &\qquad \left. + m^2\frac{A-B}{C}(\wp u - e_3) + \frac{T^2}{G^2}\right\} \\ &= \frac{M\epsilon^4\mu^2}{T}(\wp\omega - \wp u), \end{aligned}$$

provided that

$$\wp\omega = \frac{\dfrac{B-C}{A}e_1 + \dfrac{C-A}{B}e_2 + \dfrac{A-B}{C}e_3 - \dfrac{T^2}{m^2G^2}}{\dfrac{B-C}{A} + \dfrac{C-A}{B} + \dfrac{A-B}{C}},$$

so that

$$\begin{aligned} \wp\omega - e_1 &= -\frac{1}{\mu^2}\left(\frac{G}{B} - \frac{T}{G}\right)\left(\frac{G}{C} - \frac{T}{G}\right) \\ &= -\frac{1}{\mu^2}(\beta-\delta)(\gamma-\delta), \end{aligned}$$

with Hermite's notation, *Sur quelques applications*, &c., page 24.

Similarly, by symmetry,

$$\wp\omega - e_2 = -\frac{1}{\mu^2}\left(\frac{G}{C}-\frac{T}{G}\right)\left(\frac{G}{A}-\frac{T}{G}\right)$$

$$= -\frac{1}{\mu^2}(\gamma-\delta)(\alpha-\delta),$$

$$\wp\omega - e_3 = -\frac{1}{\mu^2}\left(\frac{G}{A}-\frac{T}{G}\right)\left(\frac{G}{B}-\frac{T}{G}\right)$$

$$= -\frac{1}{\mu^2}(\alpha-\delta)(\beta-\delta).$$

Our ω here is therefore practically the same as the ω employed by Hermite, p. 27; also

$$\wp'^2\omega = -4\frac{(AT-G^2)^2(BT-G^2)^2(CT-G^2)^2}{A^2B^2C^2G^6\mu^6},$$

so that, now (Routh, *Rigid Dynamics*, § 532),

$$\cot^2\zeta = \frac{\frac{M\epsilon^4 T}{G^2}}{\frac{M\epsilon^4\mu^2}{T}(\wp\omega-\wp v)} = \frac{T^2}{G^2\mu^2}\frac{1}{\wp\omega-\wp u},$$

and

$$\frac{di\phi}{du} = \lambda + i\frac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCG^3\mu^3}\frac{1}{\wp\omega-\wp u}$$

$$= \lambda - \tfrac{1}{2}\frac{\wp'\omega}{\wp u-\wp\omega},$$

on putting

$$\wp'\omega = 2i\frac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCG^3\mu^2}.$$

26. Since the product $(AT-G^2)(BT-G^2)(CT-G^2)$ is negative, $\wp'\omega$ is a negative imaginary, and therefore $\omega = s\omega_3$, where s is a real proper fraction; and then, by comparison with Hermite's v, p. 27, $s = v/K'$; also v is the same as the quantity denoted by a in the article "Solution of Euler's equations of motion by means of Elliptic Functions" (*Quarterly Journal of Mathematics*, Vol. XIV., p. 267), and a slight consideration will easily show how the angles λ, μ, ν of that article can be expressed, by means of Weierstrass's notation, in as simple a manner as the angle ϕ of the herpolhode. The projection on the invariable plane of the spherical curve described by a point fixed on one of the principal axes will then be found to be given by an

equation of the form

$$x+iy = k\,\frac{\sigma(u+v)}{\sigma u\,\sigma v}\exp(\lambda-\zeta v)\,u,$$

where $\omega+v=\omega_1$, $\omega_1+\omega_3$, or ω_3, according as the point is on the principal axis of A, B, or C.

27. De Sparre's theorem (*Comptes Rendus*, t. 99 and 101), which asserts that the herpolhode has *no* points of inflexion, can now be proved; for, writing the equation of the herpolhode

$$x+iy = \phi u = \frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\quad\lambda-\zeta\omega)\,u,$$

$$x-iy = \phi_1 u = \frac{\sigma(u-\omega)}{\sigma u\,\sigma\omega}\exp(-\lambda+\zeta\omega)\,u,$$

then $$\frac{d^2\phi}{du^2} = x''+iy'', \quad \frac{d\phi_1}{du} = x'-iy;$$

and therefore $$\frac{d^2\phi}{du^2}\,\frac{d\phi_1}{du} = x'x''+y'y''+i(x'y''-x''y'),$$

and therefore the imaginary part is zero at a point of inflexion.

But $$\frac{d\phi}{du} = \{\zeta(u+\omega)-\zeta u-\zeta\omega+\lambda\}\,\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\lambda-\zeta\omega)\,u$$

$$= \left(\tfrac{1}{2}\,\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}+\lambda\right)\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\lambda-\zeta\omega)\,u,$$

$$\frac{d^2\phi}{du^2} = \left(2\wp u+\wp\omega+\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}\lambda+\lambda^2\right)\frac{\sigma(u+\omega)}{\sigma u\,\sigma\omega}\exp(\lambda-\zeta\omega)\,u,$$

$$\frac{d\phi_1}{du} = \left(\tfrac{1}{2}\,\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}-\lambda\right)\frac{\sigma(u-\omega)}{\sigma u\,\sigma\omega}\exp(-\lambda+\zeta\omega)\,u;$$

and therefore

$$\frac{d^2\phi}{du^2}\,\frac{d\phi_1}{du} = -\left(2\wp u+\wp\omega+\frac{\wp' u-\wp'\omega}{\wp u-\wp\omega}\lambda+\lambda^2\right)\left\{\tfrac{1}{2}(\wp' u+\wp'\omega)-\lambda(\wp u-\wp\omega)\right\};$$

and the imaginary part of this expression, since $\wp u$, $\wp' u$, and $\wp\omega$ are real, $\wp'\omega$ is negative imaginary, and λ is positive imaginary, is

$$\left(2\wp u+\wp\omega-\frac{\lambda\,\wp'\omega}{\wp u-\wp\omega}+\lambda^2\right)\left\{\tfrac{1}{2}\wp'\omega-\lambda(\wp u-\wp\omega)\right\}+\frac{\lambda\,\wp'^2 u}{\wp u-\wp\omega};$$

and this, equated to zero, gives

$$(\wp u+\tfrac{1}{2}\wp\omega)\,\wp'\omega-\lambda\,(2\,\wp^2 u-\wp u\,\wp\omega-\wp^2\omega)$$
$$+\tfrac{1}{2}\frac{\wp'^2 u-\wp'^2\omega}{\wp u-\wp\omega}\lambda+\tfrac{3}{2}\lambda^2\wp'\omega-\lambda^3(\wp u-\wp\omega)=0,$$

reducing to a simple equation for $\wp u$, giving

$$\wp u=\frac{3\lambda^2\wp'\omega+\lambda\,(6\,\wp^2\omega-g_2)+\wp\omega\,\wp'\omega}{2\lambda^3-6\lambda\wp\omega-2\,\wp'\omega},$$

and, since $\wp u-e_2$ will be found to be negative, $\wp u-e_3$ positive, it follows that the points of inflexion are imaginary, although they may be real on Sylvester's generalized herpolhodes, described by the point of contact of a confocal ellipsoid rolling on a parallel plane.

28. Let us consider a third dynamical problem, the determination of the curve assumed by a uniform chain fixed at two points of a rotating body, when in relative equilibrium.

The equations of relative equilibrium, with the usual notation, are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+m\omega^2 x=0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right)+m\omega^2 y=0,$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right)=0,$$

the axis of z being the axis of rotation.

Three first integrals of these equations are immediately obtainable, namely,

$$T\frac{dz}{ds}=T_0,$$

$$Tr^2\frac{d\theta}{ds}=h,$$

$$T+\tfrac{1}{2}m\omega^2 r^2=\lambda,$$

r and θ denoting polar coordinates in a plane perpendicular to the axis of rotation.

Therefore

$$\frac{ds}{dz}=\frac{T}{T_0},$$

$$r^2\frac{d\theta}{dz}=\frac{h}{T_0},$$

and therefore $$1+\frac{dr^2}{dz^2}+r^2\frac{d\theta^2}{dz^2}=\frac{T^2}{T_0^2},$$

or $$1+\frac{dr^2}{dz^2}+\frac{h^2}{T_0^2 r^2}=\frac{T^2}{T_0^2},$$

or $$\frac{r^2 dr^2}{dz^2}=\frac{T^2}{T_0^2}r^2-r^2-\frac{h^2}{T_0^2},$$

or $$\left(\frac{dr^2}{dz}\right)^2=\frac{4r^2}{T_0^2}\left(\tfrac{1}{2}m\omega^2 r^2-\lambda\right)^2-4r^2-\frac{4h^2}{T_0^2},$$

a cubic function of r^2; and therefore r^2 is an elliptic function of z.

29. If the chain is fixed to two points lying in a plane through the axis of rotation, the chain will lie altogether in this plane, and the equation of its curve takes the simple form with Jacobi's notation

$$r/a=\operatorname{sn} zK/c.$$

In the general case, however, the chain sweeps out a surface of revolution, whose equation is

$$x^2+y^2=a^2\operatorname{cn}^2 zK/c+b^2\operatorname{sn}^2 zK/c,$$

$2a$ and $2b$ being the maximum and minimum diameters of the surface.

30. Employing Weierstrass's notation, we must put

$$r^2=k^2(\wp u-\wp v),$$

so that $$k^4\wp'^2 u\left(\frac{du}{dz}\right)^2=\frac{m^2\omega^4k^6}{T_0^2}(\wp u-\wp v)^3$$

$$-\frac{4\lambda m\omega^2k^4}{T_0^2}(\wp u-\wp v)^2+\left(\frac{4\lambda^2}{T_0^2}-4\right)(\wp u-\wp v)-\frac{4h^2}{T_0^2},$$

$$=\tfrac{1}{4}\frac{m^2\omega^4k^6}{T^2}(4\wp^3 u-g_2\wp u-g_3),$$

provided that $$\frac{du^2}{dz^2}=\tfrac{1}{4}\frac{m^2\omega^4k^2}{T_0^2},$$

and $\wp v$, g_2, and g_3 are suitably chosen.

Then, since $u=v$ make $r^2=0$, therefore

$$k^4\wp'^2 v\left(\frac{du}{dz}\right)^2=-\frac{4h^2}{T_0^2},\quad\text{or}\quad \wp'^2 v=-\frac{16h^2}{m^2\omega^4k^6};$$

and $\wp' v$ is consequently imaginary.

31. From the equation $$r^2 \frac{d\theta}{dz} = \frac{h}{T_0},$$

we obtain $$\frac{d\theta}{du} = \frac{h}{T_0 r^2} \frac{dz}{du} = \frac{2h}{m\omega^2 k^3} \frac{1}{\wp u - \wp v},$$

or $$\frac{di\theta}{dz} = -\tfrac{1}{2} \frac{\wp' v}{\wp u - \wp v},$$

if $\wp' v$ is taken as negative imaginary, and consequently $v = s\omega_3$, where s is a real proper fraction.

Then $$i\theta = \tfrac{1}{2} \log \frac{\sigma(u+v)}{\sigma(u-v)} - u\zeta v,$$

or $$e^{i\theta} = \sqrt{\left\{ \frac{\sigma(u+v)}{\sigma(u-v)} \right\}} e^{-u\zeta v}.$$

But $$r^2 = k^2 (\wp u - \wp v),$$

and therefore

$$x + iy = re^{i\theta} = ik \frac{\sigma(u+v)}{\sigma u \, \sigma v} \exp(-u\zeta v) = ik\,\phi(u, v),$$

$$x - iy = ik\,\phi(-u, v),$$

give the equations of the curve assumed by the chain, the projection of which on a plane perpendicular to the axis of revolution is consequently similar to a herpolhode or the projection of the motion of the bob of a spherical pendulum; also, du/dz being constant, we must put

$$u = z\omega_1 / c + \omega_3,$$

where $2c$ denotes the whole length of a complete wave of the chain, and the constant ω_3 is added in order to make $\wp u$ oscillate in value between e_2 and e_3.

When the chain is fixed at two points in a plane perpendicular to the axis of rotation, the curve formed by the chain will be a plane curve, given by an equation of the form

$$x + iy = \frac{\sigma(u+v)}{\sigma u \, \sigma v} \exp(-u\zeta v) = \phi(u, v),$$

the general equation of a catenary under a central force varying as the distance.

A particular case of these catenaries is the curve

$$r \cosh m\theta = a,$$

a Cotes's spiral, the separating herpolhode.

When this catenary is a free orbit under a central force P, the intensity must vary as Ar^3+Br, partly as the third and partly as the first power of the distance, and then we shall find that du/dt is constant; in fact,

$$\frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = \frac{\ddot{x}+i\ddot{y}}{x+iy} = \frac{P}{r} = 2\wp u+\wp v = 2r^2+3\wp v;$$

so that

$$P = 2r^3+3r\wp v.$$

32. These three dynamical illustrations show that the elliptic integral of the third kind must be treated as the logarithm of the vector of the corresponding azimuthal motion.

[The solutions of other well-known dynamical problems can be exhibited in a similar form.

Thus, in the motion of a particle inside a smooth paraboloid of revolution with vertical axis, the projection of the path on a horizontal plane is given by

$$x+iy = k\phi\,(u, v)\, e^{\lambda u},$$

where

$$\frac{dt}{du} = e_1-\wp u,$$

and

$$x^2+y^2 = 4mz = k^2\,(\wp v-\wp u).$$

In the motion of a particle on a smooth vertical cone of semivertical angle α,

$$x+iy = k\,(\wp v-\wp u)\left\{\frac{\sigma\,(u+v)}{\sigma\,(u-v)}\,\exp\,(-2u\zeta v)\right\}^{\operatorname{cosec}\alpha},$$

$$r = k\,(\wp v-\wp u),$$

and

$$\frac{dt}{du} = \wp v-\wp u;$$

and the motion of a sphere rolling on a rough vertical cone will be given by similar equations.

The path of a particle on a smooth horizontal table, attached to one end of a string passing through a hole in the table and supporting a particle hanging vertically, is of a similar nature; and the particle may be supposed to move on a smooth vertical cone, the string passing through a hole at the vertex and supporting another particle, without material alteration of the preceding results.

The formulæ obtained by M. Halphen in the *Journal de l'Ecole Polytechnique*, 1884, "Sur une courbe élastique," for the solution of

M. Maurice Lévy's problem of the curve formed by a plate, originally cylindrically circular, due to a uniform pressure on one side (for instance, a circular tube exposed to an external collapsing pressure), can now be slightly modified in order to be expressed in a similar form; for, writing $2v$ and $u-v$ for his v and u, then

$$x+iy = a\phi(u+v, 2v)\,\phi(u-v, 2v),$$

$$r^2 = a^2\{\wp(u+v)-\wp 2v\}\{\wp(u-v)-\wp 2v\},$$

$$Ar^4+Br^2+C = \tfrac{1}{2}a\{\wp(u+v)+\wp(u-v)-2\wp 2v\},$$

$$(Ar^4+Br^2+C)^2-r^2 = \tfrac{1}{4}a^2\{\wp(u-v)-\wp(u+v)\}^2.$$

These equations are derived from M. Lévy's relation

$$\frac{1}{\rho} = 4Ar^2+2B,$$

connecting ρ the radius of the curvature, and r the distance from a fixed point of any point on the curve; and the relation is immediately obtained by taking moments about any point on the curve,

$$\frac{a}{\rho} = -\frac{1}{\wp' 2v}\,\frac{\wp'(u-v)-\wp'(u+v)}{\wp(u-v)-\wp(u+v)},$$

and then

$$\frac{ds}{du} = ia\wp' 2v.$$

De Sparre's theorem (§ 27) has been considered recently by numerous writers; for instance, Darboux, Mannheim, Resal in the *Journal de l'Ecole Polytechnique*, 1885, and Hess, in the *Math. Annalen.*]

April 7th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Messrs. Joseph Edwards, M.A., Fellow of Sidney Sussex College, Cambridge; Robert Russell, M.A., Trinity College, Dublin; and A. N. Whitehead, B.A., Fellow and Lecturer of Trinity College, Cambridge, were elected members.

The following communications were read (or taken as read):

On the Intersections of a Circle and a Plane Curve: Prof. Genese.

A new Theory of Harmonic Polygons: Rev. T. C. Simmons.

On some Properties of Simplicissima, with especial regard to the related spherical Loci: W. J. C. Sharp.

On Briot and Bouquet's Theory of the Differential Equation $F\left(u, \frac{du}{dz}\right) = 0$: Prof. Cayley, F.R.S.

The Cosine Orthocentres of a Plane Triangle and a Cubic through them: R. Tucker.

Note on a Tetrahedron: Dr. Wolstenholme.

The following presents were received:

"Mathematical Questions and their Solutions, from the 'Educational Times,'" Vol. XLVI.

"Proceedings of the Royal Society," Vol. XLII., Nos. 251 and 252.

"Educational Times," for April.

"Transactions of the Cambridge Philosophical Society," Vol. XIV., Part II., 1887.

"Johns Hopkins University Circulars," Vol. VI., No. 56, March, 1887.

"Smithsonian Report," 1884, Part II.; Washington, 1885.

"Transactions of the Connecticut Academy," Vol. VII., Part I.; Newhaven, 1886.

"Bulletin des Sciences Mathématiques," T. XI., March and April, 1887.

"Bulletin de la Société Mathématique de France," T. XV., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," B. XI., St. 2 and 3.

"Acta Mathematica," 9: 3.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., F. 3 and 4; Febbrajo, 1887.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Num. 29 and 30; Firenze.

"Actes de la Société Helvétique des Sciences Naturelles," réunie à Genève les 10, 11, et 12 Août 1886, 8vo; compte-rendu 1885—86; Genève, 1886.

"Compte-rendu des Travaux présentés à la Soixante-neuvième Session de la Société Helvétique des Sciences Naturelles," réunie à Genève les 10, 11, et 12 Août 1886, 8vo; Genève, 1886.

"Mittheilungen der Naturforschenden Gesellschaft in Bern aus dem Jahre 1886," Nr. 1143—1168, 8vo; Bern, 1887.

"Jahrbuch über die Fortschritte der Mathematik," B. XVI., H. 2; Jahrgang 1884.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VII., No. 4; Coimbra, 1886.

"Ueber die Integration der m-ten Wurzel aus einer rationalen Function," von F. Klitzkowski, 8vo pamphlet; Greifswald, 1887. Inaugural dissertation.)

A new Method for the Investigation of Harmonic Polygons.

By Rev. T. C. Simmons, M.A.

[*Read April 7th*, 1887.]

1. If $P_1P_2P_3P_4 \ldots$ be a cyclic polygon such that the perpendiculars drawn on the sides from some internal point K are all respectively proportional to those sides, then $P_1P_2P_3P_4 \ldots$ is said to be a *harmonic polygon* whose *Lemoine point* is K.

The properties of a harmonic *quadrilateral* were first investigated by Mr. Tucker (*Educational Times Reprint*, Vol. XLIV., pp. 125—135), and continued by Prof. Neuberg in *Mathesis*, Tom. v., pp. 202, 217, 241, 265. The only writer who has hitherto dealt with the general harmonic polygon of n sides is Dr. Casey (*Proc. R. I. Academy*, 2nd Series, Vol. IV., No. 5, pp. 355, 356; and *Sequel to Euclid*, 4th Edition, pp. 199—206).

The object of the present paper is to show that an extension of my "Theorem in Conics" (*Proceedings*, Vol. XVII., p. 418) affords a more convenient and searching mode than those hitherto adopted for investigating the properties of these figures.

2. *Any harmonic polygon can be orthogonally projected into another polygon all of whose sides subtend the same angle at the projection of the Lemoine point.*

This our fundamental theorem can be proved as follows. Let the tangents to the circumcircle at any two alternate vertices P_1, P_3 of the polygon intersect in V. Then the ratio of the perpendiculars from V on P_1P_2, P_2P_3 is

$$\frac{P_1V}{P_3V}\frac{\sin VP_1P_2}{\sin VP_3P_2} = \frac{\sin VP_1P_2}{\sin VP_3P_2} = \frac{\sin P_1P_3P_2}{\sin P_3P_1P_2} = \frac{P_1P_2}{P_3P_2},$$

which is also the ratio of the perpendiculars from K on P_1P_2, P_2P_3; hence KP_2V is a straight line. Let now the circle $P_1P_2P_3\ldots$ be orthogonally projected into an ellipse $p_1p_2p_3\ldots$ having k the projection of K for focus, and let the projection of V be v; kp_2v is still a straight line, and the angles p_1kv, p_3kv are equal. Hence p_1p_2, p_2p_3, and consequently p_3p_4, p_4p_5, ..., all subtend the same angle at k.

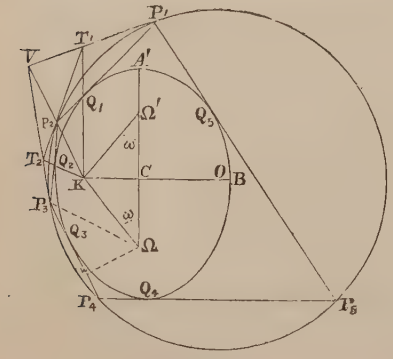

Fig. 1.

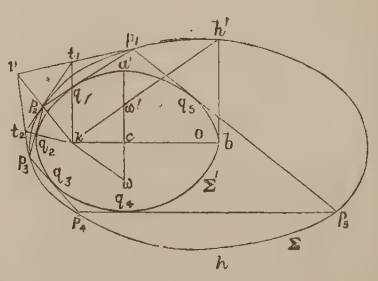

Fig. 2.

3. *To any harmonic polygon of n sides can be inscribed an ellipse, known as its Brocard ellipse. If n be odd, the lines joining the vertices to the points of contact of the ellipse with the opposite sides are all concurrent; if n be even, the lines joining the points of contact of the ellipse with pairs of opposite sides, as also the diagonals of the polygon, are all concurrent. In each case the point of concurrence coincides with the Lemoine point of the polygon.*

This follows immediately from the fact that, since p_1p_2, p_2p_3, p_3p_4 ... (Fig. 2) all subtend the same angle at the focus, they all touch another ellipse $q_1q_2q_3$ ... having the same focus and corresponding directrix, and the points of contact q_1, q_2, q_3 ... lie on the bisectors of the angles p_1kp_2, p_2kp_3, p_3kp_4 ... .

We may conveniently denote the ellipse $p_1p_2p_3$ ... by Σ, and the ellipse $q_1q_2q_3$ ... by Σ'. If the respective eccentricities be then denoted by ϵ, ϵ', and the *latera recta* by 2λ, $2\lambda'$, we have

$$\epsilon' = \epsilon \cos\frac{\pi}{n}, \quad \lambda' = \lambda \cos\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1, 2).$$

Again, let the semi-axes of Σ, Σ' (Fig. 2), and the Brocard ellipse (Fig. 1), be denoted respectively by (a, β), (a', β'), and (a, b). The relation between the three sets of axes is

$$\frac{aa'}{b\beta'} = \frac{a}{\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

for a', β' are the respective projections of b, a, and the cosine of the angle of projection $= \beta/a$.

The relation between the three eccentricities is hence

$$(1-e^2)(1-\epsilon'^2) = 1-\epsilon^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Moreover, it is evident that the minor axis of the Brocard ellipse is collinear with the line joining O, the circumcentre of $P_1P_2P_3$ ..., with its Lemoine point K; also that the two polars of K with respect to circumcircle and Brocard ellipse are identical, for they both project into the common directrix of Σ, Σ'.

4. *The poles of the sides of a harmonic polygon with respect to the circumcircle lie on a conic which we will call the polar conic of the polygon. One of its principal axes is collinear with OK, and the three polars of K with respect to circumcircle, Brocard ellipse, and polar conic are all coincident.*

This is evident from the fact that t_1, t_2, t_3 ..., the poles of p_1p_2, p_2p_3, p_3p_4 ..., all lie on a conic having the same focus and corres-

ponding directrix as Σ, and whose eccentricity and semi-latus rectum are respectively

$$\epsilon \sec \frac{\pi}{n} \quad \text{and} \quad \lambda \sec \frac{\pi}{n}.*$$

Since kt_1, kt_2, kt_3 ... bisect the angles p_1kp_2, p_2kp_3, p_3kp_4 ..., respectively, it is seen that the lines joining K to the vertices of the polar polygon $T_1T_2T_3$ ... pass through the points of contact of the sides of $P_1P_2P_3$ ... with the Brocard ellipse $Q_1Q_2Q_3$ ... .

5. *If the Brocard ellipse of a harmonic polygon be orthogonally projected into a circle, its points of contact with the sides project into the vertices of another harmonic polygon whose Lemoine point is the projection of K. If the polar conic of a harmonic polygon be orthogonally projected into a circle, the polar polygon projects into another harmonic polygon whose Lemoine point is the projection of K.*

We have here only to observe that, in Fig. 2, the points q_1, q_2, q_3 ... all lie on an ellipse whose focus is k, at which q_1q_2, q_2q_3, q_3q_4 ... all subtend equal angles. A like remark applies to the points t_1, t_2, t_3 ... ; in this latter case, the original circumcircle of course projects into the Brocard ellipse of the new harmonic polygon.

6. In Fig. 2 it is clear that q_1q_2, q_2q_3, q_3q_4 ... also touch an ellipse internal to Σ'. If the points of contact be denoted by r_1, r_2, r_3 ..., then r_1r_2, r_2r_3, r_3r_4 ... also touch another ellipse, and so on *ad infinitum*. Again, if at t_1, t_2, t_3 ... tangents be drawn to the conic $t_1t_2t_3$ ..., they form a polygon $u_1u_2u_3$ ... inscribed in a conic external to $t_1t_2t_3$ ... ; this gives rise to another similarly derived polygon inscribed in a conic, and so on. Thus $t_1t_2t_3$ ..., Σ, $q_1q_2q_3$ ... (or Σ') may be considered merely as three members of a system of conics infinite in number, and all having the same focus and directrix; their eccentricities and latera-recta each forming geometrical progressions, whose common ratio, proceeding inwards, is $\cos \frac{\pi}{n}$. The limiting form of the inner-

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* The trilinear equation of the *polar conic* of a triangle, determined from the fact that it passes through $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, and that the polar with respect to it of (a, b, c) must be $\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0$,

is $$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}+3\left(\frac{\beta\gamma}{bc}+\frac{\gamma\alpha}{ca}+\frac{\alpha\beta}{ab}\right)=0.$$

I find the coordinates of its centre to be proportional to

$$a(3b^2+3c^2-5a^2), \quad b(3c^2+3a^2-5b^2), \quad c(3a^2+3b^2-5c^2).$$

most conic is a point-circle at k; that of the outermost is a right line coinciding with the common directrix.

Reverting now to the original harmonic polygon $P_1P_2P_3\ldots$, we see that there exist in connexion with it an infinite number of other polygons, both internal and external, each inscribed in a conic, one of whose principal axes lies along OK. The ultimate form of the innermost conic is an infinitesimal ellipse situated at K, having its major axis perpendicular to KO, and the ratio of whose axes is β/α, the cosine of the angle of projection. The ultimate form of the outermost conic is a line coinciding with the polar of K, which is the same for every conic of the series. If any one of the conics is orthogonally projected into a circle, the corresponding polygon is projected into a harmonic polygon whose Lemoine point is the projection of K; the adjacent internal conic is at the same time projected into its Brocard ellipse, and the adjacent external conic into its polar conic. It is, moreover, clear that any two alternate conics are so related that either of them is the polar reciprocal of the other with respect to the intervening conic. The vertices of the polygons lie alternately on one or other of two fixed systems, each of n lines, diverging from K; in other words, any two alternate polygons have K for their centre of perspective.

It is not, of course, possible to orthogonally project into circles more than a limited number of the *external* conics, since they quickly become hyperbolas.

7. *The perpendicular from the Lemoine point of a harmonic polygon on the Lemoine line is the harmonic mean of the perpendiculars drawn on the same line from the vertices of the polygon.*

[Note.—The Lemoine line of a harmonic polygon is the common polar of the Lemoine point with respect to the circumcircle, Brocard ellipse, polar conic, and the other conics mentioned in Art. 6.]

In Fig. 2, the polar equation of Σ,

$$\rho\,(1-e\cos\theta) = l,$$

gives

$$\frac{l}{kp_1}+\frac{l}{kp_2}+\frac{l}{kp_3}+\ldots+\frac{l}{kp_u} = \{1-e\cos\mu\}+\left\{1-e\cos\left(\mu+\frac{2\pi}{n}\right)\right\}+\ldots$$
$$\ldots+\left\{1-e\cos\left(\mu+\frac{2n\pi-2\pi}{n}\right)\right\},$$

where μ is the angle which kp_1 makes with the major axis. That is to say,

$$\frac{1}{kp_1}+\frac{1}{kp_2}+\frac{1}{kp_3}+\ldots+\frac{1}{kp_n}=\frac{n}{l},$$

so that, if p', p'', ... $p^{(n)}$ be the perpendiculars from p_1, p_2, ... p_n on the directrix, and if p be the same perpendicular from k,

$$\frac{1}{p'}+\frac{1}{p''}+\frac{1}{p'''}+\ldots+\frac{1}{p^{(n)}}=\frac{n}{p},$$

and the like property (with regard to the Lemoine line) holds in the circle from which the ellipse is derived by projection.

8. *If Ω, Ω' be the foci of the Brocard ellipse, the produced directions of $K\Omega$, $K\Omega'$ intersect the circumcircle on the tangent at either extremity of the minor axis.*

For, if the tangent at either vertex b of Σ' intersect Σ in h and h' (Fig. 2), and kh, kh' meet aca', the minor axis of Σ', in ω, ω', we have $\angle hkh'=2\pi/n$, so that

$$c\omega'=ck\tan\frac{\pi}{n}=\epsilon'.a'.\tan\frac{\pi}{n}.$$

But, from (1) and (4),

$$(1-e^2)(1-\epsilon'^2)=1-\epsilon'^2\sec^2\frac{\pi}{n},$$

giving
$$e^2=\frac{\epsilon'^2\tan^2\frac{\pi}{n}}{1-\epsilon'^2}=\frac{(a'.\epsilon')^2\tan^2\frac{\pi}{n}}{\beta'^2};$$

therefore
$$c\omega'=e.\beta'=e.ca',$$

showing that, when ca' is transformed into the major axis of the Brocard ellipse, ω' transforms into a focus.

9. The angle $K\Omega\Omega'$ or $K\Omega'\Omega$ is called the *Brocard angle* of the harmonic polygon, and is commonly denoted by ω.* The circumradius of the polygon is denoted by R.

We proceed to express in terms of R and ω the eccentricities and the lengths of the axes and *latera recta* of Σ and Σ'.

For this purpose we may conveniently consider Σ as the projection of its own auxiliary circle, so that

$$a=R,\quad a'=b\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5,6).$$

But, if HH' denote that chord of the circumcircle whose projection is

* The reader will easily distinguish between the ω of Fig. 1, which denotes an *angle*, and the ω of Fig. 2, which denotes a *point*.

hh', we have

$$\cot\omega = \frac{\frac{1}{2}HH'}{KB} = \frac{a}{\beta}\cdot\frac{h'b}{kb} = \frac{\tan\pi/n}{\sqrt{(1-\epsilon^2)}},$$

therefore $$\epsilon^2 = 1-\tan^2\omega\tan^2\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

therefore $$\beta = a\sqrt{(1-\epsilon^2)} = R\tan\omega\tan\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and $$\lambda = \beta^2/a = R\tan^2\omega\tan^2\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Again, $$\lambda' = \lambda\cos\frac{\pi}{n} = R\tan^2\omega\tan\frac{\pi}{n}\sin\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots(10),$$

$$\epsilon'^2 = \epsilon^2\cos^2\frac{\pi}{n} = \cos^2\frac{\pi}{n} - \tan^2\omega\sin^2\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots(11).$$

But $$\frac{\lambda'}{a'} = \frac{\beta'^2}{a'^2} = 1-\epsilon'^2 = \sin^2\frac{\pi}{n}\sec^2\omega,$$

therefore $$a' = \lambda'\cos^2\omega\,\mathrm{cosec}^2\frac{\pi}{n} = R\sin^2\omega\sec\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots(12),$$

and $$\beta' = a'\sin\frac{\pi}{n}\sec\omega = R\sin^2\omega\sec\omega\tan\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots(13).$$

10. We can now express the eccentricity and axes of the Brocard ellipse in terms of R and ω.

For $$b = a' = R\sin^2\omega\sec\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14),$$

hence, from (3), (5), (8), (13),

$$a = \frac{ba\beta'}{a'\beta} = \frac{a\beta'}{\beta} = R\sin\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

giving $$e^2 = 1-\sin^2\omega\sec^2\frac{\pi}{n} \equiv \cos^2\omega - \sin^2\omega\tan^2\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots(16).$$

11. *The points Ω, Ω' lie on the circle described on OK as diameter, called the Brocard circle of the polygon. The ratio of $O\Omega$ to the circumradius is equal to the eccentricity of the Brocard ellipse.*

For, if ψ denote the angle of projection,

$$\cos\psi = \beta/a = \tan\omega\tan\frac{\pi}{n};$$

therefore, if KM perpendicular to OK meet the circumcircle in M,

$$KM = \lambda \sec\psi = R \tan\omega \tan\frac{\pi}{n}, \text{ from (9)},$$

therefore $$OK^2 = R^2\left(1-\tan^2\omega\tan^2\frac{\pi}{n}\right) = R^2e^2\sec^2\omega \text{ * } \ldots\ldots\ldots(17).$$

Also $$K\Omega = \Omega C\sec\omega \text{ (see Fig. 1)} = ae\sec\omega,$$

and $$KC = \Omega C\tan\omega = ae\tan\omega,$$

therefore $$KO\,.\,KC = Re\sec\omega\,.\,ae\tan\omega$$
$$= a^2e^2\sec^2\omega \text{ by (15)} = K\Omega^2,$$

therefore $K\Omega O$ is a right angle,

therefore $$\angle KO\Omega = \omega \text{ and } O\Omega = OK\cos\omega = eR.$$

12. It will be necessary now to distinguish between the two foci Ω, Ω' of the Brocard ellipse, which are called the *Brocard points* of the harmonic polygon. That focus Ω which comes next after K in revolving in a positive sense round the triangle $K\Omega\Omega'$ we will call the *positive* Brocard point, and the other focus Ω' the *negative* Brocard point.†

If $P_1P_2P_3P_4P_5\ldots$ be a harmonic polygon, such that P_1, P_2, $P_3\ldots$ are placed consecutively round the polygon in the positive direction of rotation, and if Ω, Ω' be the respectively positive and negative Brocard points, then the angles ΩP_1P_2, ΩP_2P_3, $\Omega P_3P_4\ldots$, as also the angles $\Omega' P_2P_1$, $\Omega' P_3P_2$, $\Omega' P_4P_3\ldots$, are each equal to ω.

For, draw ΩW perpendicular on P_3P_4 (Fig. 1), then $OP_3 = R$, $CW = R\sin\omega$, by (15), $= OP_3\sin\omega$, also $C\Omega = \Omega O\sin\omega$, therefore

$$CW : C\Omega = OP_3 : O\Omega.$$

But, if P be taken on WP_3 such that $\angle WP\Omega = \omega$, we have, by similar triangles, $C\Omega O$, $W\Omega P$,

$$C\Omega : \Omega W = O\Omega : \Omega P,$$

---

* The lengths of OK and $\Omega\Omega'$ have been given also by Dr. Casey (*Sequel*, p. 203, Cor. 6, and p. 217, Ex. 92).

† These terms, like most of the others here used, are derived by extension from the geometry of the triangle, reference to which will be found in the sequel. The positive and negative points have been recently, in France, denominated *point direct* and *point rétrograde;* we have not adopted them, as the term retrograde seems hardly admissible, in English, for a point not in motion.

also the angles $C\Omega W$, $O\Omega P$ are equal, therefore

$$CW : C\Omega = OP : O\Omega,$$

or P coincides with P_3.

Hence $\angle \Omega P_3 P_4 = \omega = \angle \Omega P_1 P_2 = \angle \Omega P_2 P_3 = \ldots$;

and in like manner for the angles

$$\Omega' P_2 P_1,\ \Omega' P_3 P_2 \ldots .$$

13. *The length of the perpendicular from the Lemoine point on any side P_3P_4 of the polygon is equal to $\frac{1}{2} P_3P_4 \tan \omega$.* The angles subtended by P_3P_4 at Ω and Ω' are the supplements of the adjacent angles of the polygon.*

Join ΩP_3, $\Omega' P_4$, and let them intersect in U. The angle

$$\Omega U \Omega' = P_3 U P_4 = \pi - 2\omega = \Omega K \Omega',$$

thus U lies on the Brocard circle, and OUK is a right angle. But OU is perpendicular to P_3P_4, since the triangle UP_3P_4 is isosceles;

thus

distance of K from P_3P_4 = distance of U from $P_3P_4 = \frac{1}{2} P_3P_4 \tan \omega$.

Again,

$$\angle P_3 \Omega P_4 = \pi - \Omega P_3 P_4 - (P_3 P_4 P_5 - \Omega P_4 P_5) = \pi - P_3 P_4 P_5 \text{ (Art. 12)};$$

and in like manner $\angle P_3 \Omega' P_4 = \pi - P_2 P_3 P_4$.

14. *The product of any two alternate sides of a harmonic polygon divided by the product of the sines of the two intermediate angles is constant, and equal to $R^2 \sec^2 \frac{\pi}{n}$; and the product of all the sides divided by the product of the sines of all the angles is equal to $R^n \sec^n \frac{\pi}{n}$.*

For the length of the perpendicular ΩW (Fig. 1) on any side P_3P_4 is

$$\Omega P_3 \sin \omega = \frac{P_2P_3 \sin \Omega P_2 P_3}{\sin P_2 \Omega P_3} \sin \omega = \frac{P_2P_3 \sin^2 \omega}{\sin P_3} \text{ (Arts. 12 and 13)};$$

---

* Dr. Casey, who does not mention the properties of the preceding article, adopts this as the *definition* of ω (*Sequel*, p. 199.)

the length of the perpendicular from Ω' on P_3P_4 is, in like manner,

$$\frac{P_5P_4 \sin^2 \omega}{\sin P_4}.$$

But the product of these perpendiculars is equal to the square on the semi-minor-axis of the Brocard ellipse, that is, by (14), to

$$R^2 \sin^4 \omega \sec^2 \frac{\pi}{n},$$

therefore

$$\frac{P_2P_3 \cdot P_4P_5}{\sin P_3 \sin P_4} = R^2 \sec^2 \frac{\pi}{n},$$

from which the latter part of the proposition also readily follows.

15. From the preceding article many interesting results may be deduced. For instance, it is at once seen that in a harmonic quadrilateral *the rectangles contained by the two pairs of opposite sides are equal*, a theorem originally due to Mr. Tucker. (*Reprint*, Vol. XLIV., p. 127.)

Again, since $2R \sin P_3 = P_2P_4$, $2R \sin P_4 = P_3P_5$,

we have

$$\frac{P_2P_3 \cdot P_4P_5}{P_2P_4 \cdot P_3P_5} = \tfrac{1}{4} \sec^2 \frac{\pi}{n}.$$

By forming the six equations of this type in the case of the harmonic hexagon, and multiplying, we deduce that *the product of the six sides of two cosymmedian triangles* is equal to* 27 *times the product of the sides of the corresponding harmonic hexagon.*

Again, by Euc. VI., D,

$$P_2P_4 \cdot P_3P_5 = P_2P_3 \cdot P_4P_5 + P_3P_4 \cdot P_2P_5,$$

so that in the hexagon $2P_2P_3 \cdot P_4P_5 = P_3P_4 \cdot P_2P_5$. From the six equations of this type we deduce, by multiplication, that *the product of the squares of the diagonals of a harmonic hexagon is equal to* 64 *times the product of the sides.*

16. The Brocard angle ω of $P_1P_2P_2 \ldots$ can be expressed in terms of the sides and area. For, denoting the latter by S, we have by Art. 13,

$$2S = \tfrac{1}{2} \tan \omega \, (P_1P_2^2 + P_2P_3^2 + P_3P_4^2 + \ldots);$$

---

* If, in any harmonic polygon, P_1K, P_2K, $P_3K \ldots$ be produced to meet the circumcircle again in P_1', P_2', $P_3' \ldots$, a new polygon $P_1'P_2'P_3' \ldots$ is found, which is evidently harmonic and has the same Brocard ellipse, Lemoine point, and Brocard angle as $P_1P_2P_3 \ldots$ (see Figs. 1, 2); the two polygons have been called by Dr. Casey *cosymmedian*. It is also evident that the polygon of $2n$ sides determined by the $2n$ vertices is harmonic.

that is, *the tangent of the Brocard angle of a harmonic polygon is equal to four times the area of the polygon divided by the sum of the squares of the sides.*

Another expression for ω can be given in terms of the areas of the triangles $P_1P_2P_3$, $P_2P_3P_4$, $P_3P_4P_5$ ... . For, denoting the latter respectively by Δ_2, Δ_3, Δ_4 ..., we have

$$\text{area of } \Omega P_3P_4 = \tfrac{1}{2}\Omega W \cdot P_3P_4 = \tfrac{1}{2} P_3P_4 \cdot P_2P_3 \operatorname{cosec} P_3 \sin^2\omega \text{ (by Art. 14)}$$
$$= \Delta_3 \operatorname{cosec}^2 P_3 \sin^2\omega.$$

Whence, by the addition of all such equations, we obtain

$$\Delta_1 \operatorname{cosec}^2 P_1 + \Delta_2 \operatorname{cosec}^2 P_2 + \Delta_3 \operatorname{cosec}^2 P_3 + \ldots + \Delta_n \operatorname{cosec}^2 P_n = S \operatorname{cosec}^2 \omega.$$

17. *If on P_1P_2, P_2P_3, P_3P_4 ... we take points D_1, D_2, D_3 ... so that P_1D_1, P_2D_2, P_3D_3 ..., all measured in the same vectorial direction, subtend the same angle θ at Ω, the points D_1, D_2, D_3 ... determine a harmonic polygon similar to $P_1P_2P_3$ ..., the radius of whose circumcircle is $R \sin\omega \operatorname{cosec}(\theta+\omega)$.*

For ΩD_1, ΩD_2, ΩD_3 ... are inclined at the same angle θ to $\Omega P_1, \Omega P_2, \Omega P_3$ ..... respectively, and are to them in the constant ratio

$$\Omega D_1 : \Omega P_1 = \sin\omega : \sin(\theta+\omega);$$

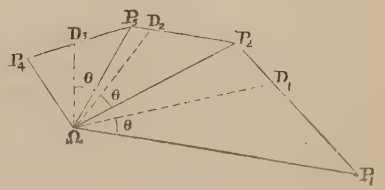

Fig. 3.

also P_1, P_2, P_3 ... all lie on a circle of radius R.

$D_1D_2D_3$ ... may be called a *Tucker polygon*, and its circumcircle a *Tucker circle*, of the original polygon $P_1P_2P_3$ ... . The positive Brocard point Ω of the latter is also the positive Brocard point of all its Tucker polygons (see Art. 12), and hence is their *permanent centre of similitude*, in sense used by Neuberg (*Proceedings*, Vol. XVI., p. 186).

18. If θ_1, θ_2 be two values of θ, such that $\theta_1+\theta_2 = \pi-2\omega$, so that $\sin(\theta_1+\omega) = \sin(\theta_2+\omega)$, the two corresponding Tucker circles are equal. The *minimum* Tucker circle will be the auxiliary circle of the Brocard ellipse, corresponding to the value $\theta = \frac{1}{2}\pi-\omega$, in which case ΩD_1, ΩD_2, ΩD_3 ... are respectively perpendicular to P_1P_2, P_2P_3, P_3P_4 .... In Fig. 3 we have measured the angles θ all in the *positive* vectorial direction, with reference to ΩP_1, ΩP_2, ΩP_3 ... ; if they are all measured in the opposite direction, so that D_1, D_2 ... lie respectively on the produced directions of P_2P_1, P_3P_2 ..., then θ must be considered negative. We thus see that the circles occur in pairs, and their areas are capable of all values between that of the

auxiliary circle of the Brocard ellipse and ∞. The mode of constructing the two Tucker circles of any given area greater than $\pi R^2 \sin^2 \omega$ is obvious.

19. *As θ varies, any point X invariably connected with the polygon $D_1D_2D_3$ ... describes a straight line, and any straight line XY invariably connected with it envelopes a parabola having Ω for focus.*

The proof is identical with that given in Prof. Neuberg's Paper, "Sur les figures semblablement variables," in *Proceedings*, Vol. XVI., p. 186.

If d_1, d_2, d_3 ... be the feet of the perpendiculars from Ω on P_1P_2, P_2P_3, P_3P_4 ..., then $d_1d_2d_3$ ... is the minimum Tucker polygon of the system. Let ΩZ be drawn perpendicular on XY, and let x, y, z be the positions of X, Y, Z for this latter polygon. It is clear, from what has been established, that *the locus of X is a line through x perpendicular to Ωx, and the envelope of XY a parabola whose focus is Ω and whose vertex is z.* In particular, the sides D_1D_2, D_2D_3 ... of the Tucker polygons envelope parabolas having Ω for focus, and d_1, d_2 ... respectively for vertices.

20. *The centres of the Tucker circles of a harmonic polygon all lie on the minor axis of its Brocard ellipse, and the distance of the centre of any circle from Ω is equal to its radius multiplied by the eccentricity of the same ellipse.*

The first part is obvious, from the fact that the centre C of the Brocard ellipse is the circumcentre of the minimum Tucker polygon $d_1d_2d_3$ ..., and therefore the centres all lie on the line through C perpendicular to ΩC. The last part follows from Art. 11 ;* for, if O' be the centre, and R' the radius of the circle $D_1D_2D_3$ ..., we have $O'\Omega = e'R'$, where e' is the eccentricity of the Brocard ellipse of $D_1D_2D_3$ .... But, $D_1D_2D_3$ ... being similar to $P_1P_2P_3$ ..., the Brocard ellipses are similar ; thus $e' = e$.

21. *The Brocard ellipse of a harmonic polygon is the envelope of its Tucker circles.*

For it is well known that the distance of any point on the minor axis of an ellipse from the focus is equal to the normal drawn from that point multiplied by the eccentricity. Hence the radius of any Tucker circle is equal to the normal drawn from its centre to the Brocard ellipse.

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* This particular mode of deducing Art. 20 from Art. 11 is shorter than was my own originally ; and is due to a suggestion of Mr. R. F. Davis, who had also communicated to me his calculations (by the method of invariants) of the Brocard-axes of the quadrilateral and hexagon before I arrived at the general formulæ myself.

22. In Fig. 3, the sides of $D_1D_2D_3\ldots$ are inclined to the corresponding sides of $P_1P_2P_3\ldots$, all at the same angle θ. If the circle $D_1D_2D_3\ldots$ meet P_3P_2 in D_3' and P_2P_1 in D_2', the angle $P_2D_2'D_3' = P_2D_2D_1$ in the same segment; thus the sides of $D_1'D_2'D_3'\ldots$ are inclined to the corresponding sides of $P_1P_2P_3\ldots$, all at the same angle θ, and $D_1'D_2'D_3'\ldots$ is similar to $P_1P_2P_3\ldots$. Thus every Tucker circle determines by its intersections with the sides of the fundamental polygon *two* similar harmonic polygons, or, in other words, two Tucker polygons, one of which (since the angles $D_1'\Omega'P_1$, $D_2'\Omega'P_2$, $D_3\Omega'P_3\ldots$ are easily seen all to equal θ) possesses with regard to Ω' the same properties which the other possesses with regard to Ω.

23. When $\theta = \omega$, the radius of the Tucker circle is $\frac{1}{2}R\sec\omega$, and the distance of its centre from Ω is $\frac{1}{2}Re\sec\omega$, which is the radius of the Brocard circle (Art. 11), so that the two circles are concentric. This particular Tucker circle is called, from the analogy of the triangle, the *first Lemoine circle* of $P_1P_2P_3\ldots$. In this case,

$$P_2D_2 = \frac{\Omega P_2\sin\omega}{\sin 2\omega} = \tfrac{1}{2}\Omega P_2\sec\omega = \frac{\frac{1}{2}P_2P_1\sin\omega}{\sin P_2\Omega P_1}\sec\omega = \frac{\frac{1}{2}P_2P_1\tan\omega}{\sin P_2};$$

but $P_2D_2\sin P_2$ is equal to the perpendicular from D_2 on P_1P_2; and (Art. 13) $\frac{1}{2}P_1P_2\tan\omega$ is equal to the perpendicular from K on P_1P_2; thus KD_2 is parallel to P_1P_2. Similarly, KD_3' is parallel to P_3P_4; thus the lines joining K to the two points of intersection of the circle with any side of the polygon are parallel to the two adjacent sides.

Again, when $\theta = \frac{1}{2}\pi$, the radius of the circle is $R\tan\omega$, and its centre is at K. This Tucker circle is known as the *second Lemoine circle*, or the *cosine circle*, of the polygon, the latter denomination being derived from the fact that the intercept made by it on any side of the polygon is proportional to the cosine of the angle which that side subtends at the circumference of the circumcircle (*Sequel to Euclid*, 4th Edition, p. 203).*

24. From Figs. 1 and 2, it is evident that, if *any* point be taken on the circumcircle of $P_1P_2P_3\ldots$, a harmonic polygon can be drawn having that point for one of its vertices, and having the same Brocard ellipse and Lemoine point. We will venture to call those polygons of the same number of sides which have the same Brocard ellipse (and, in consequence, the same circumcircle and Lemoine point) *co-Brocardal*

* This property, already signalised by M. Lemoine in the case of the *triangle*, was first perceived to hold for the harmonic *quadrilateral* by Mr. M'Cay, to whom also occurred the first glimpse of the existence of harmonic polygons (see *Educational Times Reprint*, Vol. XLIV., p. 135).

*polygons.** Since, corresponding to a given ellipse, there exists only one system of circles having double contact with the curve and with their centres on the minor axis, it follows that *all co-Brocardal polygons have the same system of Tucker circles.* They all have, moreover, *the same first Lemoine and cosine circles*, since the radii and the positions of the centres of those circles depend in each case on R and ω only.

25. Dr. Casey has shown (*Sequel*, p. 200) that, if the alternate vertices $P_1, P_3, P_5 \ldots$ of a harmonic polygon be joined, another harmonic polygon is formed having K for Lemoine point; and that the same thing is true for the polygons $P_1P_4P_7P_{10}\ldots$, $P_1P_5P_9P_{13}\ldots$, and so on. These results are at once deducible from Figs. 1 and 2, where it will be seen that the sides of the projected polygons subtend the respective angles $\frac{4\pi}{n}, \frac{6\pi}{n}, \frac{8\pi}{n}\ldots$ at k.

Generally speaking, if every m^{th} vertex of a harmonic polygon of n sides be taken, a new harmonic polygon is determined. When m is a sub-multiple of n, the new polygon closes up at the end of the first complete revolution round the circle. When m is not a sub-multiple of n, we must revolve round the circle more than once; and when m is prime to n, m revolutions will be necessary before the polygon closes. All the formulæ of the preceding articles will here still apply, by interpreting the new polygon as having a *fractional* number $\frac{n}{m}$ of sides.

We may even generalise further, and say that *any* chord P_1P_2 of a circle can be considered as the side of a harmonic polygon having *any* given internal point K as Lemoine point. For, project the circle into an ellipse Σ having k, the projection of K, for focus, and let ψ be the angle subtended at k by p_1p_2 the projection of P_1P_2; put $\frac{2\pi}{\psi} \equiv n$; then, if n be an integer, P_1P_2 is the side of an ordinary harmonic polygon of n sides. If n be a commensurable fraction $\frac{r}{s}$ in its lowest terms, the polygon does not close till after s revolutions round the circle; if n be incommensurable, the polygon does not close at all. In any case the projection of the Brocard ellipse is that ellipse, having the same focus and corresponding directrix as Σ, which touches p_1p_2.

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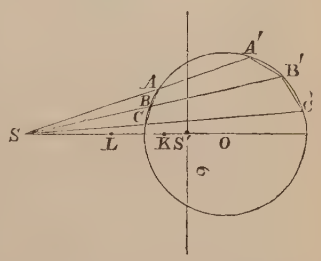
* The theory of co-Brocardal triangles will be found worked out in Chap. vii. of my article *On the recent Geometry of the Triangle*, in a forthcoming work by the Rev. J. J. Milne, M.A., entitled "Companion to Weekly Problem Papers."

26. It may be added that, if every vertex of a harmonic polygon be joined to the two alternate vertices, the polygon having P_1P_3, P_2P_4, P_3P_5 ... for *consecutive* sides is inscribed in one ellipse and circumscribed to another, both ellipses having their major axes collinear with OK. If the outer ellipse be projected into a circle, the polygon is projected into a harmonic polygon having the projection of K for Lemoine point, and the inner ellipse is at the same time projected into its Brocard ellipse. A corresponding conclusion holds if every vertex of $P_1P_2P_3$ ... be joined to the third, or fourth, or fifth ... vertex to the right and left. In fact, propositions of this kind, and many others, are so obvious on inspection of Figs. 1 and 2, that it is hardly necessary to refer to them.

27. Dr. Casey has also shown that the vertices of a harmonic polygon may be considered as the inverses of the vertices of a regular polygon of the same number of sides. The following proof, by *harmonic projection*, which has been communicated to me by Prof. Neuberg, may be of interest to those who prefer not to employ the methods of inversion.

(1) The harmonic projection of a curve with respect to a given *centre* S, and a given *axis* σ of perspective, both in the plane of the curve, is found as follows. Let A be any point on the curve, and let SA meet σ in s. Then, if A' be taken on SA such that $[SAsA']$ is harmonic, A' is called the *harmonic projection* of A, and similarly for other points on the curve. It is evident that, for all centres and axes of perspective, concurrent straight lines are harmonically projected into concurrent straight lines, and a tangent to a curve into a tangent to the projected curve.

(2) Let AB, BC be two consecutive sides of a harmonic polygon whose Lemoine point is K. Produce OK to meet the polar of K in L, mark off LS or LS' equal to the tangent from L to the circle; then σ, the polar of S, will pass through S'. It is evident that, with S as centre and σ as axis of perspective, any point A on the circle harmonically projects into the point A', where SA again meets the circle; also that K harmonically projects into O. Now the tangents at A and C will meet on KB produced;



hence the tangents at A' and C' will also meet on OB' produced; that is to say, $A'B' = B'C'$.

Hence, with S (or S') as centre, and the corresponding polar as axis of perspective, the harmonic polygon $ABC\ldots$ projects harmonically into a regular polygon $A'B'C'\ldots$; which is the same as saying that the vertices of the one invert into the vertices of the other, with either S or S' for centre of inversion.

The connexion between a harmonic and a regular polygon can also be seen by reciprocating, with respect to k, the ellipse Σ of Fig. 2 into a circle, in which case $p_1 p_2 p_3 \ldots$ reciprocates into a regular polygon.

[To the memoirs cited in Art. 1 must now be added another by MM. Tarry and Neuberg, published by the *Association française pour l'avancement des sciences* (*Congrès de Nancy, séance du* 13 *août*, 1886), entitled "Sur les polygones et les polyèdres harmoniques," and which contains interesting and ingenious extensions of some of the foregoing properties to the case of solid figures, by the method of harmonic projection. It may here be mentioned that, although the *géométrie brocardienne* of the triangle was only *re*discovered and amplified in England, having previously originated on the continent (see the brief historical sketch on pp. 180–184 of "Companion to Weekly Problem Papers"), all the fundamental properties of harmonic polygons are of British origin. As to the present paper, the theorems of Arts. 2, 4, 5, 6, 7, 14, 15 (last two paragraphs), 16, 19, 24, 26, are new; and, of the remaining propositions and formulæ, many had been previously assumed by extension from the triangle, without rigorous demonstration.]

On Relations between Circles and Algebraic Curves, with Applications to Dynamics. *By* Prof. R. W. Genese, M.A.

[*Read April 7th*, 1887.]

1. Let
$$u_n + u_{n-1} + u_{n-2} + \&c. = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
be the equation to a curve of the n^{th} degree, where u_r, as usual, denotes a homogeneous expression of r dimensions in x, y.

Let $P(x, y)$ be one of the points of intersection of (1) with a circle of radius ρ about the origin O; θ the inclination of OP to the axis of x. Also, let
$$\lambda = \rho(\cos\theta + \iota \sin\theta),$$

then
$$\frac{\rho^2}{\lambda} = \rho(\cos\theta - \iota\sin\theta),$$

$$\left.\begin{aligned}\frac{1}{2}\left(\lambda + \frac{\rho^2}{\lambda}\right) &= \rho\cos\theta = x\\ \frac{1}{2\iota}\left(\lambda - \frac{\rho^2}{\lambda}\right) &= \rho\sin\theta = y\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Substituting these values of x, y in (1), we shall obtain, *in general*, an equation of degree $2n$ in λ.

The sum of the roots will be $\Sigma\rho\cos\theta + \iota\Sigma\rho\sin\theta$, *i.e.*, the real part will be the sum of the $2n$ abscissæ of intersections, the imaginary part the sum of the ordinates.

The product of the roots will be, by De Moivre's theorem,
$$\rho^{2n}(\cos\Sigma\theta + \iota\sin\Sigma\theta).$$

Now, let
$$u_n = ax^n + bx^{n-1}y + cx^{n-2}y^2 + \ldots,$$
$$u_{n-1} = a_1x^{n-1} + b_1x^{n-2}y + \ldots,$$
&c.

Then, using (2), and multiplying by λ^n, we get

$$\frac{1}{2^n}\left\{a(\lambda^2+\rho^2)^n + \frac{b}{\iota}(\lambda^2+\rho^2)^{n-1}(\lambda^2-\rho^2) + \ldots\right\}$$
$$+\frac{\lambda}{2^{n-1}}\left\{a_1(\lambda^2+\rho^2)^{n-1} + \frac{b_1}{\iota}(\lambda^2+\rho^2)^{n-2}(\lambda^2-\rho^2) + \ldots\right\}$$
$$+ \&c. = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Whence,
$$\text{sum of roots} = -2\,\frac{a_1 + b_1\iota^{-1} + c_1\iota^{-2} + \ldots}{a + b\iota^{-1} + c\iota^{-2} + \ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and
$$\text{product of roots} = \rho^{2n}\,\frac{a - b\iota^{-1} + c\iota^{-2} - \&c.}{a + b\iota^{-1} + c\iota^{-2} + \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

2. From (4) we infer that

> "*The centre of mean position of the* $2n$ *points of intersection of any algebraic curve with a circle is independent of the radius of the circle if its centre be given.*" ……(A).

This theorem (which admits of simpler proof) was given by M. d'Ocagne in the *Nouvelles Annales de Mathématiques* for 1886, page 295; and a more general theorem by Liouville, in the *Journal de Mathématiques pures et appliquées*, 1841.

3. From (5) it is clear that $\cos\Sigma\theta$ and $\sin\Sigma\theta$ depend only on the

coefficients of u_n, which do not change with change of origin, and therefore that

The sum of the inclinations to a given axis of the radii vectores common to a given algebraic curve and a circle is independent both of the radius of the circle and of the position of its centre. .....(B).

4. Since, in general, u_n and u_{n-1} determine the asymptotes of (1), we see that the point in theorem (A), and the sum of angles in (B), may be obtained by using the asymptotes in place of the curve (1). Both theorems are obvious geometrically in the case in which (1) degenerates into straight lines. With respect to (B), it may be noted that, though pairs of lines given by $u_n = 0$ may be imaginary, yet the bisectors of the angles between them are real.

5. Let the centre of mean position for circles of centre O be denoted by $G\ (\bar{x}, \bar{y})$. Then there is a simple connection between O and G which may be thus investigated.

Let the origin be moved to the point $O'\ (\alpha, \beta)$; then u_n is unaltered and u_{n-1} is changed by a linear function of α, β, and the constants a, b, c, &c. Thus, if $(\bar{x}', \bar{y}')$ be the new position of G referred to the *old* axes,

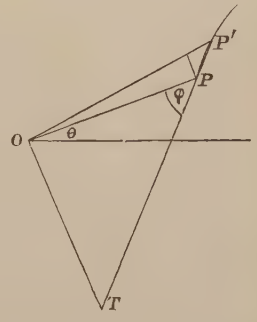
$$\left.\begin{aligned} \bar{x}' &= \alpha + \bar{x} + p\alpha + q\beta \\ \bar{y}' &= \beta + \bar{y} + r\alpha + s\beta \end{aligned}\right\} \text{.....(6)}$$

where p, q, r, s are constants.

Therefore α, β are expressible linearly in terms of $\bar{x}'$, $\bar{y}'$, and *the locus of G is of the same degree as the locus of O.*

6. Now, let the radius ρ receive a small increment $d\rho$, and in consequence let the point of intersection P move to P'. By (B), $\Sigma\theta$ is constant, therefore

$$\Sigma\, d\theta = 0 \text{.....(7).}$$



Also ρ and $d\rho$ are the same for all the points P, therefore

$$\Sigma \frac{\rho\, d\theta}{d\rho} = 0,$$

i.e.,

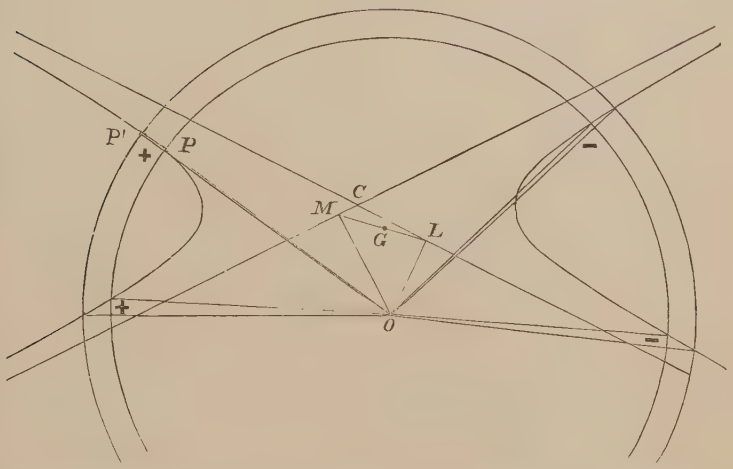
$$\Sigma \tan\phi = 0 \text{.....(8),}$$

where ϕ is the angle the tangent to the given curve at P makes with ρ.

Thus, if OT be the polar subtangent at P,

$$\Sigma OT = 0.$$

Also
$$\Sigma \rho^2 d\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$



Therefore the sum of such areas as POP', taken with due regard to sign, is zero. Since this is true for consecutive small changes in ρ, and therefore θ, it is clearly true for finite changes in ρ.*

In using the theorem, it must be observed that, if ρ become a tangent to the curve in the passage from P to P', then part of the area POP' is negative and part positive, according to the variation of θ.

7. Again, by theorem (A), Σx is constant, therefore

$$\Sigma dx = 0,$$

therefore
$$\Sigma \frac{dx}{ds} \cdot \frac{ds}{d\rho} = 0,$$

i.e.,
$$\Sigma \cos\psi \sec\phi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

ψ being the angle which the tangent PT' makes with the axis of x.

This result may be expressed in a simple geometrical form. Let the normal at P meet the axis of x in G. Then

$$\cos\phi = \sin OPG, \quad \cos\psi = \sin OGP,$$

* A remarkable memoir by M. G. Humbert, in the current *Journal de Math. pures, &c.* (1887), "*Sur le théorème d'Abel*," contains (page 354) more general theorems.

hence $$\Sigma \frac{\sin OGP}{\sin OPG} = 0,$$

i.e., $$\Sigma \frac{OP}{OG} = 0,$$

or, since OP is the same for all,

$$\Sigma \frac{1}{OG} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

It is worthy of remark that the axis of x is any line whatever through O. The polygon formed by the normals at the $2n$ points P has then the property that the polar line of O with respect to it is the line at infinity. Thus if the equation to the sides be

$$(1-a_1x-b_1y)(1-a_2x-b_2y) \ldots = 0,$$

we have $$\Sigma a = 0, \text{ and } \Sigma b = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

This property has been noticed by Liouville.

8. The ordinary method of reciprocation applied to the preceding theorem, yields interesting results. Thus we have, from (B),

If common tangents be drawn to a curve of any class and a variable circle of given centre, the sum of their inclinations to any fixed axis is constant. ..... (C).

Care must be taken to consider the direction of a tangent as that leaving the circle in the positive sense of rotation round the centre. It is clear that the constant is twice the sum of the inclinations of the tangents from the concentric point-circle.

9. From (8),

The sum of the reciprocals of the common tangents to a curve and a circle, with due regard to sign, is zero. ..... (D),

A few words of explanation may be added, viz., the corresponding ϕ's of reciprocal curves are equal, and $\tan\phi = \frac{p}{t}$, where p is the radius of the circle and t the common tangent. Or, the theorem may be deduced independently of (8); thus, if (p, α) be the polar coordinates of the point of contact of a common tangent with the circle, by (C),

$$\Sigma\alpha = \text{constant},$$

therefore $$\Sigma d\alpha = 0,$$

therefore $$\Sigma \frac{da}{dp} = 0,$$

therefore $$\Sigma \frac{1}{t} = 0.*$$

From (10)... If θ be the vectorial angle of the point of contact of a common tangent with the curve,

$$\Sigma \sin\theta \sec\phi = 0 \quad \text{.....(13).}$$

From (11)... If OT be the polar subtangent of the point of contact of a common tangent with the curve

$$\Sigma OT = 0 \text{.....(14).}$$

10. The preceding theory holds, as stated, in general. If, however, the curve of n^{th} degree be circular, or the curve of n^{th} class have a focus at the centre of the variable circle, the theorems either fail or need modification. For,

1°. If the curve (1) be simply circular, then u_n contains the factor x^2+y^2 or ρ^2, and its terms become mixed in the equation for λ with those of u_{n-2}. In fact, if

$$u_n = (x^2+y^2)(a'x^{n-2}+b'x^{n-3}y+\&c.),$$

the equation (3) is replaced by

$$\frac{1}{2^{n-1}}\{a_1(\lambda^2+\rho^2)^{n-1}+b_1(\lambda^2+\rho^2)^{n-2}(\lambda^2-\rho^2)+\&c.\}$$
$$+\frac{\lambda}{2^{n-2}}\{(a'\rho^2+a_2)(\lambda^2+\rho^2)^{n-2}+\&c.\}+\&c.=0.$$

The sum of the $2n-2$ values of λ now contains ρ^2, and theorem (A) fails, theorem (B) remains true for concentric circles, but $\Sigma\theta$ varies with the centre of the system.

If, however, the given curve (1) have a point-circle asymptote at O, *i.e.*, if u_{n-1} also be divisible by x^2+y^2, then both theorems fail. In this case O is a quadruple focus.

2°. If the given curve be bicircular [*i.e.*, if u_n contain $(x^2+y^2)^2$ as a factor, and u_{n-1} contain (x^2+y^2)], theorem (A) fails, and theorem (B) holds only for concentric circles.

The same conclusion holds if the curve have the circular points at

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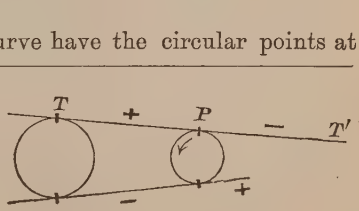
* The convention of signs may be explained thus: let $T'PT$ be a tangent to the circle, and consider PT positive and PT' negative. Now let the circle rotate in its plane round the centre. When PT touches the given curve we have a positive tangent, when PT' a negative.



infinity, I, J, as singular points of any degree of multiplicity, provided the line IJ itself do not touch the curve at I and J.

3°. If I, J be multiple points of the curve (1), and if the curve touch the line at infinity at I, J, then theorem (A) again obtains, and also theorem (B) in the modified form.

The conclusion seems to be that theorem (A) is not true for circular curves unless they touch the line at infinity; and that theorem (B) in its modified form for concentric circles is true in all cases except one, viz., when the curve is singly circular, and the centre of the cutting circle is a quadruple focus of the curve.

It follows that the reciprocal theorems (C), (D) [obtained by reciprocating (B) with respect to a circle, centre O] fail in one case only, viz., when the given curve is circular with a single focus at the centre of the variable circle.

Applications to Dynamics.

11. Theorem (A) states that, if equal weights be placed at the intersections of a curve of n^{th} degree and a circle of given centre O, the position of the weight centre G is independent of the radius of the circle. Applying the method of reciprocation (explained *Proceedings Lond. Math. Soc.*, Vol. XVII., p. 409), and using a circle of centre O, we learn that

> *If equal forces act along the* $2n$ *common tangents to a curve of the* n^{th} *class and a circle of fixed centre, the line of action of the resultant is independent of the radius of the circle.* .....(E).

12. Again, theorem (A) may be re-stated,

If the radii vectores to the points of intersection above represent forces, their resultant is independent of the radius of the circle.

Now let P, P' (see Fig. Art. 6) be intersections for radii ρ and $\rho + d\rho$, then a series of forces PP' acting at O would be in equilibrium. They would also be in equilibrium if acting along PP', &c., for

the sum of the moments round $O = 2$ sum of areas $POP' = 0$, by (9).

Also $$\frac{PP'}{d\rho} = \sec \phi.$$

Hence the following theorem,

> *If along the tangents to a curve, at its intersections with a circle, forces act proportional to the secants of the angles made with the radii, they will be in equilibrium*.....(F).

13. Reciprocating with respect to the centre of the circle, we find that

If, at the points of contact with a curve of the common tangents to the curve and any circle, weights act inversely proportional to the tangents, with their proper sign, they will balance. .....(G).

14. Again, returning to the figure of Art. 6, theorem (F) shows that a set of forces of the type PT balance.

Therefore forces represented by OT, &c. have the same resultant as forces represented by OP, &c., *i.e.*, a fixed resultant ($2n\ OG$).

Therefore $\Sigma\, OT \sin(\theta+90°)$ is constant,

or, $\Sigma\, \rho \tan\phi \cos\theta$ „ ;

so $\Sigma\, \rho \tan\phi \sin\theta$ „ ;

and $\Sigma \tan\phi = 0.$

Hence

Weights acting at the intersections of a curve and circle proportional to the polar subtangents to the curve (the centre of the circle being pole) are equivalent to a couple. .....(H).

Reciprocally,

If forces act along the common tangents to a curve and a circle inversely proportional to their lengths, their resultant passes through the centre of the circle. .....(I).

(I) may be verified by taking moments round the centre, and using (D).

15. The point O and the polygon of Art. 7 are thus related. If forces act along the perpendiculars from O on the sides, and inversely proportional to the lengths of the perpendiculars, they are in equilibrium.

For, if $ax+by=1$ be expressed in the form

$$x\cos\alpha + y\sin\alpha = p,$$

$$a = \frac{\cos\alpha}{p},$$

$$b = \frac{\sin\alpha}{p},$$

therefore, by (12), $\Sigma \frac{1}{p} \cos \alpha$ and $\Sigma \frac{1}{p} \sin \alpha$ both vanish.

Note on the Interpretation of Formula (4) *in the preceding paper.*

Consider n straight lines through the origin, making angles $\theta_1, \theta_2, \ldots \theta_n$ with the axis of x; their equation is

$$(x \sin \theta_1 - y \cos \theta_1)(x \sin \theta_2 - y \cos \theta_2) \ldots (x \sin \theta_n - y \cos \theta_n) = 0.$$

In the sinister, put $y = \iota x$; where $\iota = \sqrt{-1}$, it becomes

$$x^n (\sin \theta_1 - \iota \cos \theta_1)(\sin \theta_2 - \iota \cos \theta_2) \ldots,$$

or
$$(-\iota x)^n (\cos \theta_1 + \iota \sin \theta_1)(\cos \theta_2 + \iota \sin \theta_2) \ldots,$$

or
$$(-\iota x)^n (\cos \Sigma\theta + \iota \sin \Sigma\theta).$$

Hence, *if the result of writing* $y = \iota x$ *in the sinister of the equation to* n *straight lines be* $P + \iota Q$, and

1°. if n be even, then
$$\frac{Q}{P} = \tan \Sigma\theta,$$

2°. if n be odd,
$$-\frac{P}{Q} = \tan \Sigma\theta;$$

whence, in any case,
$$\frac{Q}{P} = \tan \left(\Sigma\theta + \frac{n\pi}{2} \right).$$

Thus, in the case of two straight lines

$$ax^2 + 2hxy + by^2 = 0,$$

we have, at once,
$$\tan (\theta_1 + \theta_2) = \frac{2h}{a-b}.$$

Again,
$$\frac{\cos \theta + \iota \sin \theta}{\cos \phi + \iota \sin \phi} = \cos (\theta - \phi) + \iota \sin (\theta - \phi).$$

With this help we are enabled to construct the *direction* of the line OG of the preceding paper, viz., if θ_1, θ_2, &c. be the inclinations to the axis of x of the asymptotes of the original curve, ϕ_1, ϕ_2, &c. those

of the asymptotes of the *first polar of* O, γ that of OG,

$$\cos\gamma+\iota\sin\gamma=\frac{a_1+b_1\iota^{-1}+c_1\iota^{-2}+\dots}{a+b\iota^{-1}+c\iota^{-2}} \text{ to a constant factor } près$$

$$=\frac{a+b\iota+c\iota^2}{a_1+b_1\iota+c_1\iota^2+\dots} \quad " \quad "$$

$$=\frac{(-\iota)^n(\cos\Sigma\theta+\iota\sin\Sigma\theta)}{(-\iota)^{n-1}(\cos\Sigma\phi+\iota\sin\Sigma\phi)} \quad " \quad "$$

$$=-\iota\left\{\cos(\Sigma\theta-\Sigma\phi)+\iota\sin(\Sigma\theta-\Sigma\phi)\right\},$$

whence $$\Sigma\theta-\Sigma\phi=\gamma+\frac{\pi}{2},$$

neglecting revolutions.

In the particular case of the conic, if we take the axis of x parallel to an axis of the conic, $\Sigma\theta$ is zero, $\Sigma\phi+\frac{\pi}{2}$ is the angle made by a perpendicular ON on the polar of O with the axis. *Hence OG and the perpendicular on the polar of O are equally inclined to an axis.*

Another form of the theory may be useful. Let

$$x\cos\alpha+y\sin\alpha-p=0,$$

$$x\cos\beta+y\sin\beta-q=0,$$

be two straight lines; *the ratio of the perpendiculars upon them from a circular point at infinity* ($x=\infty$, $y=\iota x$) is

$$\frac{\infty(\cos\alpha+\iota\sin\alpha)-p}{\infty(\cos\beta+\iota\sin\beta)-q}=\cos\overline{\alpha-\beta}+\iota\sin\overline{\alpha-\beta},$$

and *determines the angle between the lines.*

The introduction of constant multipliers will clearly not affect the *ratio* of the real and imaginary parts.

In biangular coordinates, for example, the coordinates of a circular point at infinity are simply $(\iota, -\iota)$; let

$$a\lambda+b\mu+c=0, \quad a'\lambda+b'\mu+c'=0$$

be two straight lines, the angle (δ) between them is determined by the ratio

$$\frac{a\iota+b(-\iota)+c}{a'\iota+b'(-\iota)+c'}=\frac{\{(a-b)\iota+c\}\{c'-(a'-b')\iota\}}{c'^2+(a'-b')^2},$$

therefore $$\tan\delta=\frac{c'(a-b)-c(a'-b')}{cc'+(a-b)(a'-b')}.$$

On Briot and Bouquet's Theory of the Differential Equation $F\left(u, \frac{du}{dz}\right) = 0$. *By* Professor Cayley.

[*Read April 7th*, 1887.]

I consider the theory of a differential equation of the first order, as developed in Briot and Bouquet's "Théorie des fonctions doublement périodiques et en particulier des fonctions elliptiques" (8°, Paris, 1859), but I make some substantial variations in the mode of treatment.

I remark that, writing $u = x$, $\frac{du}{dz} = y$, I make the theory to depend altogether upon that of the curve $F(x, y) = 0$; viz., the form of my result is, in order that a differential equation of the first order $F\left(u, \frac{du}{dz}\right) = 0$ may have a monotropic integral, the curve $F(x, y) = 0$ must satisfy certain conditions. Briot and Bouquet give their result (Theorem IV., p. 296) under a somewhat different form, as follows; viz., changing only the word "monodrome" into "monotrope," their theorem is:—

"Pour qu'une équation différentielle du premier ordre de la forme

$$\left(\frac{du}{dz}\right)^m + f_1(u)\left(\frac{du}{dz}\right)^{m-1} + \ldots + f_m(u) = 0$$

admette une intégrale monotrope: 1° les coefficients $f_1(u)$, $f_2(u) \ldots f_m(u)$ doivent être des polynômes entiers en u et, au plus, le premier du second degré, le second du quatrième degré, ... le dernier du degré $2m$; 2° quand pour une certaine valeur de u l'équation a une racine multiple différente de zéro, $\frac{du}{dz}$ doit rester monotrope par rapport à u; 3° quand pour une certaine valeur u_1 de u, l'équation a une racine multiple égale à zéro, le premier terme du développement de $\frac{du}{dz}$ suivant les puissances croissantes de $(u - u_1)^{1/n}$ doit avoir l'exposant $1 - \frac{1}{n}$, si cet exposant est plus petit que l'unité; 4° enfin l'équation différentielle que l'on déduit de la première en posant $u = \frac{1}{v}$, doit offrir pour $v = 0$ les mêmes caractères. — Ces conditions sont suffisantes."

I notice that this may be regarded as a statement referring to the curve $F(x, y), = y^m + f_1(x)\, y^{m-1} \ldots + f_m(x) = 0$, but that in 2° a notion of monotropy is assumed, y must be a monotropic function of x; and that 4° in effect introduces the new curve $F\left(\frac{1}{x}, -\frac{y}{x^2}\right) = 0$.

1. We have between the variables u and z a differential equation of the first order $F\left(u, \frac{du}{dz}\right) = 0$, F a rational and integral function of u and $\frac{du}{dz}$. The equation determines u as a function of z, and we wish to know whether u is a monotropic function of z. It will not be so if we have a tropical point, viz., a point $z = c$, such that in the neighbourhood thereof the value of u is given by a series $u - a = B(z-c)^\beta + \ldots$ in ascending powers of $z-c$, involving fractional powers of $z-c$. Conversely, if there is no tropical point, then u will be a monotropic function of z; but we must consider and exclude not only tropical points at a finite distance, but also the point at infinity, viz., there must not be any development $u = Bz^\beta + \ldots$, in descending powers, involving fractional powers of z.

2. A one-valued function (*eindeutige Function*, or *fonction uniforme* or *bien déterminée*) is monotropic; since there is only one value, the function must, after a circuit described by the point z, recover its original value. But, conversely, a monotropic function is one-valued, that is, no two or more valued function can be monotropic. For, suppose a two-valued function of z, having the values Z_1, Z_2, is monotropic; there is by hypothesis no tropical point, that is, no point $z = c$, such that the function Z_1, after a circuit described by the point z, instead of recovering its original value, assumes the value Z_2; and the like as to Z_2. But, this being so, it is meaningless to regard Z_1 and Z_2 as two values of the same function; they must be considered as two distinct and separate functions, each of them a one-valued monotropic function. And it thus appears that the two notions, monotropic and one-valued, are in fact equivalent.

3. In what precedes, the u, z are regarded as complex variables geometrically represented by means of two real points in their own planes respectively. But we now write $\frac{du}{dz} = y$, $u = x$, whence $F(x, y) = 0$, regarding x, y as the coordinates, real or imaginary, of a point of the curve represented by this equation. And this curve has, moreover, to be considered from a non-projective point of view; we have to distinguish between, and separately consider, finite points (x and y each of them finite), and the points at infinity (x and y either or each of them infinite). And it is moreover necessary to distinguish between, and separately consider, points on the axis of x (or say axial points), and points off the axis of x (say non-axial points). Taking, for convenience, a to denote a finite value which may be $= 0$, b a finite value which is not $= 0$ (or say, a not $= \infty$, b not $= \infty$ or 0),

we have thus the six cases, 1° (a, b); 2° $(a, 0)$; 3° (a, ∞); 4° (∞, b); 5° $(\infty, 0)$; 6° (∞, ∞). I regard also the axes of x and y as horizontal and vertical respectively, and speak of the tangent or element of the curve as horizontal, vertical, or inclined, accordingly.

4. In the neighbourhood of any given point of the curve, we have $y =$ a series of powers of x, the form of the series being different for different classes of points. I write $P(x)$ to denote a series (finite or infinite) $A+Bx+Cx^2+\ldots$, in ascending powers of x; it is to be throughout understood that the leading coefficient A is not $= 0$. Of course, $P(x-a)$, $P(x-a)^{1/n}$, $P(x^{-1})$, &c. will denote the like series $A+B(x-a)+\ldots$, $A+B(x-a)^{1/n}+\ldots$, $A+Bx^{-1}+\ldots$, etc. The coefficients after the leading coefficient A may any or all of them vanish; thus $P(x-a)^{1/n}$ extends to denote the series $P(x-a)$ of integer powers, but naturally the former notation is not used unless the series contains fractional powers. We can, by means of the symbol P, express for any given point of the curve the form of the expansion in the neighbourhood thereof; and, attaching to the point the expansion which belongs to it, we may, for instance, speak of a point $y = P(x-a)$; viz., if b is the constant term of the series, then this is a non-axial point (a, b), which is an ordinary (non-singular) point of the curve, and for which the element is not vertical; a like point with the element vertical would be a point $y = b+(x-a)^{\frac{1}{2}}P(x-a)^{\frac{1}{2}}$. Observe that, in the case of a multiple point, there is a separate expansion for each branch through the point, and in thus attaching an expansion to the point we regard the point as belonging to one of these branches; it is, in dealing with a multiple point, necessary to consider separately all the different expansions, that is, all the branches through the point. It is clear that for a point (a, b), $(a, 0)$, or (a, ∞) the expression for y depends on $P(x-a)$, or $P(x-a)^{1/n}$; while, for a point (∞, b), $(\infty, 0)$, or (∞, ∞), it depends on $P(x^{-1})$ or $P(x^{-1/n})$; viz., in the two cases respectively, we have a series in ascending powers of $x-a$, or in descending powers of x.

5. In regard to any given point of the curve, substituting for y, x their values $\frac{du}{dz}$, u, and thence forming the reciprocal of $\frac{du}{dz}$, we obtain $\frac{dz}{du} =$ a series in u; viz., this is a series in ascending powers of $(u-a)$, or in descending powers of u. Integrating, we have $z-c =$ a series in u, which series may or may not contain a logarithmic term $\log u$ or $\log(u-a)$; and, reverting the series, we obtain $u =$ a series in $(z-c)$. This result, applicable to the neighbourhood of the point $u = a$, may contain only integer powers of $z-c$, and be accordingly of the form

$u =$ a one-valued series in z, and we then say that the point on the curve is a "permissive" point; or it may contain fractional powers of $z-c$, and be accordingly of the form $u =$ a more-than-one-valued series in z; and we then say that the point on the curve is a "prohibitive" point. The necessary and sufficient condition in order that u may be a monotropic function of z is:—the points of the curve must be all of them permissive points; or, what is the same thing, there must not be any prohibitive point.

6. I form a complete table of permissive points, as follows:—

n a positive integer $= 2$ at least.
$\epsilon = 0, 1,$ or -1,

(a, b), $\quad y = P(x-a)$,

$(a, 0)$, $\quad y = (x-a)\,P(x-a)$ or $(x-a)^2\,P(x-a)$, $\qquad y = (x-a)^{1+\epsilon/n}\,P(x-a)^{1/n}$,

(a, ∞),

(∞, b), $\quad y = P(x^{-1})$,

$(\infty, 0)$,

(∞, ∞), $\quad y = xP(x^{-1})$ or $x^2P(x^{-1})$, $\quad y = x^{1+\epsilon/n}\,P\left(\frac{1}{x^{1/n}}\right)$,

viz., it is only a point (a, b), $(a, 0)$, (∞, b) or (∞, ∞) which may be permissive; a point (a, ∞) or $(\infty, 0)$ is prohibitive.

7. A point $y = P(x-a)$ is permissive. We have

$$\frac{du}{dz} = P(u-a),$$

$$\frac{dz}{du} = P(u-a),$$

$$z-c = (u-a)\,P(u-a),$$

$u-a = (z-c)\,P(z-c)$, the required result.

The several steps will be readily understood; the reciprocal of a series $P(u-a)$ is a series of the like form $P(u-a)$; integrating this in regard to u as a series in $u-a$ (no constant being added on the right-hand side), we obtain a series $(u-a)\,P(u-a)$; and finally, $z-c$ being equal to this expression, we obtain, by reversion, a like expression $u-a = (z-c)\,P(z-c)$.

We might, for convenience, have written x, u, z in place of $x-a$, $u-a$, $z-c$, respectively. The proof would have run

$$y = P(x), \quad \frac{du}{dz} = P(u), \quad \frac{dz}{du} = P(u), \quad z = uP(u), \quad u = zP(z),$$

meaning $u-a = (z-c)\,P\,(z-c)$, as above; and, in what follows, this is done accordingly.

It is worth while to make an instance in which the integration can be performed—say we consider the point (0, 1) of the curve $y = \frac{1+x+x^2}{1+2x}$, which point is $y = \frac{1+x+x^2}{1+2x} = P\,(x)$. We have

$$\frac{du}{dz} = \frac{1+u+u^2}{1+2u}, \quad \frac{dz}{du} = \frac{1+2u}{1+u+u^2},$$

$$z = \log\,(1+u+u^2), \text{ or } u+u^2 = (e^z-1),$$

giving a series $u = zP\,(z)$; or the point is a permissive point, as in the general case. We have here $u+u^2 = (e^z-1)$, viz., u is not a monotropic function of z, but this is by reason of a prohibitive point $(-\frac{1}{2}, \infty)$.

8. A point $y = (x-a)\,P\,(x-a)$ is permissive.

Writing as above u, z in place of $u-a$, $z-c$, we have

$$\frac{du}{dz} = uP\,(u), \quad \frac{dz}{du} = \frac{1}{u}\,P\,(u), \text{ or say } m\,\frac{dz}{du} = \frac{1}{u}+Pu,$$

whence

$$mz = \log u + uPu, \text{ that is, } ue^{uPu} = e^{mz}, \text{ or say } uP\,(u) = e^{mz},$$

whence $$u = e^{mz}\,P\,(e^{mz}), \text{ a one-valued series.}$$

I take a particular instance, the point (0, 0) for the curve $y = \frac{x-\frac{1}{3}x^3}{1-x^2}$;

here $$\frac{du}{dz} = \frac{u-\frac{1}{3}u^3}{1-u^2}, \text{ or } \frac{dz}{du} = \frac{1-u^2}{u-\frac{1}{3}u^3},$$

$$z = \log\,(u-\tfrac{1}{3}u^3), \text{ or } u-\tfrac{1}{3}u^3 = e^z;$$

the point is permissive. The equation just obtained shows that u is not a monotropic function of z, but this is by reason of the prohibitive points $(\pm 1, \infty)$.

9. A point $y = (x-a)^2\,P\,(x-a)$ is permissive. Here

$$\frac{du}{dz} = u^2\,P\,(u), \quad \frac{dz}{du} = \frac{1}{u^2}\,P\,(u), \quad = \frac{A}{u^2} + \frac{B}{u} + P\,(u),$$

whence $$z = -\frac{A}{u} + B\log u + uP\,(u),$$

where the logarithmic term occurs or does not occur, according as B

is not or is $= 0$. In the former case, we have

$$e^{z/B} = ue^{-A/Bu + uP(u)},$$

in the latter case $$z = -\frac{A}{u} + uP(u),$$

and in each case we can by reversion express u as a one-valued series in z. We may take as examples the two curves $y = \frac{x^2}{1-x}$ and $y = \frac{x^2}{1-x^2}$; in each case the point $(0, 0)$ is a permissive point, but, by reason of the prohibitive points $(1, \infty)$ and $(\pm 1, \infty)$, u is not a monotropic function of z.

10. As to the other forms of permissive points, the proofs depend upon those for the forms already considered. Thus

$$y = (x-a)^{1+s/n} P(x-a)^{1/n}, \quad \text{or} \quad \frac{du}{dz} = u^{1+s/n} P(u^{1/n}),$$

writing $v = u^{1/n}$, we find

$$\frac{dv}{dz} = u^{(1+s)/n} P(u^{1/n}), \ = v^{1+s} P(v),$$

that is $$\frac{dv}{dz} = P(v), \ vP(v), \text{ or } v^2 P(v);$$

in each case $v =$ one valued series in z, and thence $u = v^n =$ one-valued series in z.

Similarly $$y = P(x^{-1}), \ y = xP(x^{-1}), \ y = x^2 P(x^{-1}),$$

or say $$y = x^{1+s} P(x^{-1});$$

here $$\frac{du}{dz} = u^{1+s} P(u^{-1}),$$

or, writing $v = u^{-1}$, we have

$$\frac{dv}{dz} = u^{s-1} P(u^{-1}), \ = v^{1-} P(v);$$

viz., this is $$\frac{dv}{dz} = P(v), \ vP(v), \text{ and } v^2 P(v);$$

in each case $v =$ one-valued series in z, and thence $u = v^{-1} =$ one-valued series in z.

And finally $$y = x^{1+\alpha/n} P\left(\frac{1}{x^{1/n}}\right);$$

here $$\frac{du}{dz} = u^{1+\alpha/n} P\left(\frac{1}{u^{1/n}}\right),$$

or, writing $v = u^{-\alpha/n}$, we have

$$\frac{dv}{dz} = u^{(\alpha-1)/n} P\left(\frac{1}{u^{1/n}}\right), = v^{1-\alpha} P(v);$$

that is, $$\frac{dv}{dz} = P(v),\ vP(v),\ \text{or}\ v^2 P(v),\ \text{as before};$$

v = one-valued series in z, and thence $u = v^{-n}$ = one-valued series in z.

11. There is no difficulty in showing that every point of the curve, not being a permissive point as above, is a prohibitive point. Thus for a point (a, b), if the series for $y-b$ contain any fractional power, say, if we have $y = b+(x-a)^\alpha$,

then $$\frac{du}{dz} = b+u^\alpha,\ \frac{dz}{du} = \frac{1}{b} - \frac{1}{b^2} u^\alpha + \ldots,\ z = \frac{u}{b} - \frac{1}{(\alpha+1)b^2} u^{\alpha+1} + \ldots,$$

whence, reverting, $u = bz+$ series containing the fractional power $z^{\alpha+1}$.

Again, a point $y = (x-a)^m P(x-a)$, $m = 3$ or any higher integer value, is prohibitive. Attending only to the leading term, we have $\frac{du}{dz} = u^m$, whence $\frac{dz}{du} = u^{-m}$ or $z = u^{1-m}$, whence, reverting, $u = z^{1/(1-m)}$, which is a fractional power, and this is also the case if m be a negative integer. And similarly, in other cases, the series for u will always contain fractional powers of z.

12. In order that u may be a monotropic function of z, the points of the curve must be all of them permissive, or, what is the same thing, there must be no prohibitive point; we have to consider the geometrical signification of this condition. If we attend first to the ordinary (or non-singular) points of the curve, a non-axial point (a, b) must be a point $y = P(x-a)$, viz., there must be no such point with a vertical element.

An axial point $(a, 0)$ may be $y = (x-a) P(x-a)$, viz., the element may be inclined; or the point may be $y = (x-a)^2 P(x-a)$, viz., the element may be horizontal (but in this case there must be an ordinary or two-pointic contact only); or the point may be $y = (x-a)^{\frac{1}{2}} P(x-a)^{\frac{1}{2}}$, viz., the element may be vertical.

The point must not be (a, ∞), viz., there must be no asymptote parallel to (or coincident with) the axis of y.

The point may be (∞, b), $y = P(x^{-1})$, that is, there may be an ordinary, or osculating, asymptote parallel to the axis of x. But there is no point $(\infty, 0)$, that is, no asymptote coincident with the axis of x.

There may be points (∞, ∞), $y = xP(x^{-1})$, that is, ordinary or osculating asymptotes inclined to the axes; and there may also be points (∞, ∞), $y = x^2 P(x^{-1})$, that is, asymptotic parabolas of vertical axis.

13. We have, moreover, conditions as to the singular points; thus, every non-axial multiple point (a, b) must be a point with each branch of the form $y = P(x-a)$, that is, each branch must be an ordinary (non-cuspidal) branch, and must be non-vertical. The axial multiple or singular points $(a, 0)$ may have ordinary (non-cuspidal) branches $y = (x-a)P(x-a)$ and $y = (x-a)^2 P(x-a)$, inclined or horizontal, or else $y = (x-a)^{\frac{1}{2}} P(x-a)^{\frac{1}{2}}$, vertical; and there may also be cuspidal branches $y = (x-a)P(x-a)^{1/n}$, $y = (x-a)^{1+1/n} P(x-a)^{1/n}$, and $y = (x-a)^{1-1/n} P(x-a)^{1/n}$, inclined, horizontal, or vertical.

There must be no singular points (a, ∞); any such point is in fact excluded by the condition, no asymptote parallel to or coincident with the axis of y.

There may be multiple points (∞, b), but not $(\infty, 0)$; viz., as above, there may be asymptotes parallel to, but not coincident with, the axis of x; but in this case the several branches must be each of them an ordinary branch $y = P(x^{-1})$.

Finally, there may be multiple or singular points (∞, ∞), but each branch must be either an ordinary branch $y = xP(x^{-1})$ or $y = x^2 P(x^{-1})$, or a cuspidal branch

$$y = xP\left(\frac{1}{x^{1/n}}\right), \quad y = x^{1+1/n} P\left(\frac{1}{x^{1/n}}\right), \quad \text{or} \quad y = x^{1-1/n} P\left(\frac{1}{x^{1/n}}\right).$$

14. The enumeration seems long and difficult, but for a given curve we have in fact only to see that there are no ordinary non-axial points (a, b) of vertical element, and to further examine the finite singular points (a, b) and $(a, 0)$, and also the several infinite branches as giving ordinary or else singular points (a, ∞), (∞, b), $(\infty, 0)$, or (∞, ∞), viz., it has to be seen for each infinite branch that the expansion is of the proper form.

15. Writing the equation of the curve in the form

$$f_0(x)\, y^m + f_1(x)\, y^{m-1} + \ldots\ldots f_m(x) = 0,$$

where $f_0, f_1, \ldots f_m$ denote rational and integral functions of x, then, if a be a root of the equation $f_0(x) = 0$, there will be on the curve a point

(a, ∞), which is prohibitive; $f_0(x)$ is therefore a constant, or, taking it to be $= 1$, the form must be

$$y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

where $f_1, f_2, \ldots f_m$ denote rational and integral functions of x.

It is to be shown that the degrees of these functions are equal 2, 4, ... m at most. Supposing that the degrees are 2, 4, ... m, then for the points at infinity we find an expansion $y = Ax^\omega + \ldots$, in descending powers of x, by substitution in an equation

$$y^m + a_1 x^2\, y^{m-1} + a_2 x^4\, y^{m-2} \ldots + a^m x^{2m} = 0;$$

whence $\omega = 2$, and A is determined by an equation of the order m; there are thus m branches $y = x^2 P(x^{-1})$, corresponding to permissive points. If, however, any one of the functions f be of a higher degree, then it may be shown (but the formal proof is not easily given) that there will be at any rate a branch $y = Ax^\omega +$ &c., where ω has a value > 2, and we have thus a prohibitive point; hence the degrees must be as stated.

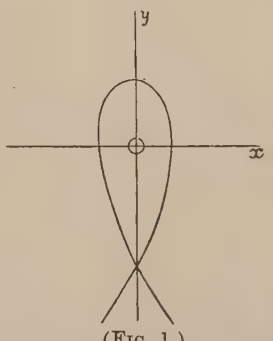
16. As an example of a differential equation with a monotropic solution, take

$$\left(\frac{du}{dz}\right)^3 + 3\left(\frac{du}{dz}\right)^2 + 27u^2 - 4 = 0;$$

we have here the curve

$$y^3 + 3y^2 + 27x^2 - 4 = 0,$$

which is a nodal cubic, as shown in the annexed figure. To put the node in evidence, the equation may be written in the form $(y+2)^2(y-1) + 27x^2 = 0$; viz., the node is the point $(0, -2)$, and the equation of the tangents is $(y+2)^2 - 27x^2 = 0$. The curve besides meets the line $x = 0$ in the point $y = 1$, where the tangent is horizontal, and it meets the axis $y = 0$ in the points $x = \pm \frac{2}{3\sqrt{3}}$, at each of which the tangent is vertical; these two points, as lying on the axis $y = 0$, are thus each of them permissive; and they are the *only* points where the tangent is vertical (in fact, the tangents from the point at infinity on the line $x = 0$ are the two lines $x = \pm \frac{2}{3\sqrt{3}}$, and the line infinity counting twice, as a tangent at an inflexion).



(Fig. 1.)

For the points at infinity, we have $y = -3x^{\frac{2}{3}} - 1 - \frac{1}{3}x^{-\frac{2}{3}} + \&c.$, which is of the form $y = x^{\frac{2}{3}} P\left(\frac{1}{x^{\frac{1}{3}}}\right)$, included in $y = x^{1-1/n} P\left(\frac{1}{x^{1/n}}\right)$, and the point is thus permissive; there is no prohibitive point, and the differential equation has a monotropic solution. This is in fact the rational solution $u = (z-c) - (z-c)^3$, or say $u = z - z^3$; the curve is thus given by the two equations $x = z - z^3$, $y = 1 - 3z^2$.

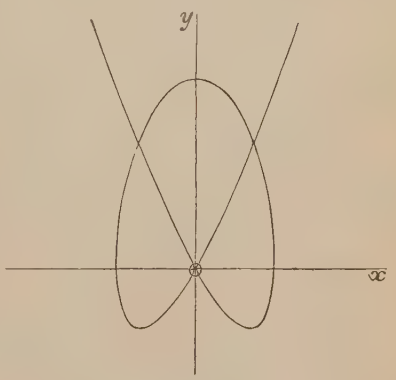
17. As a second example, take the differential equation

$$\left(\frac{du}{dz}\right)^3 - \left(\frac{du}{dz}\right)^2 + 4u^2 - 27u^4 = 0.$$

We have here the curve

$$y^3 - y^2 + 4x^2 - 27x^4 = 0,$$

which is a trinodal quartic curve, as shown in the figure. There is a node at the origin with the tangents $y^2 - 4x^2 = 0$, and, writing the equation in the form



(Fig. 2.)

$$(3y+1)(3y-2)^2 - (27x^2-2)^2 = 0,$$

we have for the other two nodes $x = \pm \frac{\sqrt{2}}{3\sqrt{3}}$, $y = \frac{2}{3}$. The lowest points are given by $x = \pm \frac{2}{3\sqrt{3}}$, $y = -\frac{1}{3}$, viz., the line $y = -\frac{1}{3}$ is a horizontal tangent. Writing $x = 0$, we have $y^3 - y^2 = 0$, that is $y^2 = 0$, the node at the origin, and $y = 1$, the height of the loop; and, writing $y = 0$, we have $4x^2 - 27x^4 = 0$, that is, $x^2 = 0$, the node at the origin, and $x = \pm \frac{2}{3\sqrt{3}}$, the other two intersections with the axis of x. The tangents at these two points are vertical; as being on the axis, they are thus permissive points. And they are the *only* points with a vertical tangent; in fact, the point $(x = 0, y = \infty)$ is a point of the curve, with the line ∞ as an osculating tangent (4-pointic intersection), hence the tangents from the point $x = 0$, $y = \infty$ are the line ∞ counting four times, and the lines $x = \pm \frac{2}{3\sqrt{3}}$ touching at the points $\left(\pm \frac{2}{3\sqrt{3}}, 0\right)$ as above. For the infinite branches we have $y = 3x + \ldots$, which is of the right form $y = x^{1+1/n} P\left(\frac{1}{x^{1/n}}\right)$. We thus

see that there is no prohibitive point, and the differential equation has a monotropic solution accordingly.

Writing z in place of $z-c$, the solution in fact is

$$u = \frac{e^z - e^{-z}}{e^z + e^{-z}} - \left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right)^3;$$

hence, putting $\theta = \frac{e^z - e^{-z}}{e^z + e^{-z}}$, the curve is given by the two equations

$$x = \theta - \theta^3, \quad y = (1-3\theta^2)(1-\theta^2), \quad = 1 - 4\theta^2 + 3\theta^4,$$

the monotropic function u which satisfies the differential equation.

18. $F\left(u, \frac{du}{dz}\right) = 0$ must be either a rational function, a singly periodic function, or a doubly periodic function of z; say the forms are

$$u = P(z), \quad u = P(e^{gz}),$$

and

$$u = P\{\mathrm{sn}(gz, k)\} + \mathrm{cn}(gz, k)\,\mathrm{dn}(gz, k)\; Q\{\mathrm{sn}(gz, k)\},$$

where P, Q denote rational functions. I do not at present consider the criteria (such as are given in Briot and Bouquet's Theorem V., p. 301) for determining by means of the curve which is the form of the integral. I remark, however, that in the first and second cases the curve is unicursal, while in the third case it is bicursal; or say that, according as the deficiency is $= 0$ or 1, the integral is rational or simply periodic, or else it is doubly periodic. Moreover, the curve being unicursal, we can express the coordinates as equal to rational functions $P(\theta)$, $Q(\theta)$ of a parameter θ; and, being bicursal, we can express them as functions of the form

$$P(\theta) + P_1(\theta)\sqrt{1-\theta^2 . 1-k^2\theta^2}, \quad Q(\theta) + Q_1(\theta)\sqrt{1-\theta^2 . 1-k^2\theta^2}$$

of the parameter θ; and supposing the coordinates (x, y), that is, u and $\frac{du}{dz}$, thus expressed, it should be easy to establish the relation of θ with z, e^{gz} or $\mathrm{sn}(gz, k)$ in the three cases respectively.

On the Properties of Simplicissima (with especial regard to the related Spherical Loci). By W. J. Curran Sharp, M.A.

[*Read April 7th*, 1887.]

I. A Simplicissimum is the figure formed in space of n dimensions by $n+1$ points, no three of which lie in a straight line, no four in a plane, no five in a hyper-plane, &c. In the question (8242) in the *Educational Times*, in which the figure was defined, Professor Sylvester gave the squared content as

$$\frac{-1}{(-2)^n\,(1.2\ldots n)^2}\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\ 1, & (1.2)^2, & 0, & (2.3)^2, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

where $(r.s)$ denotes the length of the straight line joining r^{th} and s^{th} vertices. In the solution I have deduced the result from the expression

$$\frac{1}{1.2\ldots n}\begin{vmatrix} x_1, & y_1, & z_1, & \ldots & 1 \\ x_2, & y_2, & z_2, & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n+1}, & y_{n+1}, & z_{n+1}, & \ldots & 1 \end{vmatrix}$$

for the content, x, y, and z being orthogonal coordinates. This value was assumed in analogy with

$$\frac{1}{1}\begin{vmatrix} x_1, & 1 \\ x_2, & 1 \end{vmatrix}, \qquad \frac{1}{1.2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}$$

and

$$\frac{1}{1.2.3}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

the known expressions for simplicissima in space of one, two, and three dimensions.

This value is consistent with the other view which might have been taken, that the content of the simplicissimum of the n^{th} order is $\frac{1}{n}^{\text{th}}$ of the product of the content of the simplicissimum of the $n-1^{\text{th}}$ order through n of the vertices into the perpendicular upon the linear locus through them from the remaining vertex. For

$$V = \frac{1}{n!} \begin{vmatrix} x_1, & y_1, & \ldots\ldots & 1 \\ x_2, & y_2, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_{n+1}, & y_{n+1}, & \ldots\ldots & 1 \end{vmatrix}$$

which is an absolute invariant for all orthogonal transformations.

Now take the axes so that the linear locus through all the vertices but the first shall be denoted by $x = 0$, and let the axis of x be the perpendicular upon this from the first vertex, which will be the point $p_1, 0, 0 \ldots$; therefore

$$V = \frac{1}{n!} \begin{vmatrix} p_1, & 0, & 0, & \ldots & 1 \\ 0, & y_2, & z_2, & \ldots & 1 \\ 0, & y_3, & z_3, & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix}$$

$$= \frac{p_1}{n} \times \frac{1}{(n-1)!} \begin{vmatrix} y_2, & z_2, & \ldots & 1 \\ y_3, & z_3, & \ldots & 1 \\ \ldots & \ldots & \ldots & \end{vmatrix}^{*}.$$

If, now, the centre of the spherical locus (or hyper-sphere) which can always be described through the vertices of the simplicissimum be

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* The presence of the factor $\frac{1}{n}$ may be justified in various ways, by dividing the figure contained by parallel unit linear loci into n equal and similar figures having a common vertex at the centroid; or, as pointed out by one of the referees, if V_1 be the content of the base and p the height of the simplicissimum, if sections be drawn parallel to the base at distances x and $x + \delta x$ from it, and v be the volume of the frustum between these,

$$V = \Sigma v = \int_0^p V_1 \frac{x^{n-1}}{p^{n-1}} dx = \frac{1}{n} V_1 p \ldots$$

I am indebted to the same source for a reference to another proof by Dr. Schaefli, *Quarterly Journal of Mathematics*, Vol. II., p. 279.

taken as origin of rectangular coordinates,

$$x_r^2+y_r^2+z_r^2 \ldots = R^2,$$

$$2\,(x_r x_s+y_r y_s+z_r z_s \ldots) = x_r^2+y_r^2+z_r^2 \ldots + x_s^2+y_s^2+z_s^2 \ldots$$

$$-\{(x_r-x_s)^2+(y_r-y_s)^2+(z_r-z_s)^2 \ldots\}$$

$$= 2R^2-(r\,.\,s)^2,*$$

where R is the radius of the spheric (or hyper-sphere).

Therefore

$$V^2 = \frac{1}{(n\,!)^2} \begin{vmatrix} 1, & 0, & 0, & 0, & \ldots\ldots & 0 \\ 1, & x_1, & y_1, & z_1, & \ldots\ldots & 1 \\ 1, & x_2, & y_2, & z_2, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 1, & x_{n+1}, & y_{n+1}, & z_{n+1}, & \ldots\ldots & 1 \end{vmatrix} \times \begin{vmatrix} 0, & 0, & 0, & 0, & \ldots\ldots & 1 \\ -1, & x_1, & y_1, & z_1, & \ldots\ldots & 1 \\ -1, & x_2, & y_2, & z_2, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ -1, & x_{n+1}, & y_{n+1}, & z_{n+1}, & \ldots\ldots & 1 \end{vmatrix}$$

$$= \frac{1}{(n\,!)^2} \begin{vmatrix} 0, & 1, & 1, & 1 \\ -1, & x_1^2+y_1^2+\&c., & x_1x_2+y_1y_2+\&c., & x_1x_3+y_1y_3+\&c. \ldots \\ -1, & x_1x_2+y_1y_2+\&c., & x_2^2+y_2^2+\ldots, & x_2x_3+y_2y_3+\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

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* From the above equations, if $(\bar{x}, \bar{y}, \ldots)$ be the centroid of equal weights at the vertices of the simplicissimum,

$$(n+1)\,\bar{x} = x_1+x_2+x_3 \ldots,$$

$$(n+1)\,\bar{y} = y_1+y_2+y_3 \ldots,$$

$$\&c., \qquad \&c.,$$

therefore $(n+1)^2\,\{\bar{x}^2+\bar{y}^2+\ldots\}$

$$= (x_1^2+y_1^2+\ldots)+(x_2^2+y_2^2+\ldots)+\&c.$$

$$+2\,(x_1x_2+y_1y_2+\ldots)+2\,(x_2x_3+y_2y_3+\ldots)+\&c.$$

$$= (n+1)\,R^2+\{2R^2-(1\,.\,2)^2\}+\{2R^2-(2\,.\,3)^2\}+\&c.$$

$$= (n+1)^2\,R^2-\{(1\,.\,2)^2+(2\,.\,3)^2+\&c.\};$$

so that, if $\bar{a}^2 = \bar{x}^2+\bar{y}^2+\ldots$,

$$R^2 = \bar{a}^2+\frac{1}{(n+1)^2}\{(1\,.\,2)^2+(2\,.\,3)^2+(3\,.\,1)^2+\ldots\}.$$

$$= \frac{1}{(n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ -1, & R^2, & R^2 - \frac{(1.2)^2}{2}, & R^2 - \frac{(1.3)^2}{2} & \dots \\ -1, & R^2 - \frac{(1.2)^2}{2}, & R^2, & R^2 - \frac{(2.3)^2}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{1}{2^{n+1}(n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ -2, & 0, & -(1.2)^2, & -(1.3)^2, & \dots \\ -2, & -(1.2)^2, & 0, & -(2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{-1}{(-2)^n (n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

and, since V vanishes when the $n+1$ points lie on a linear locus, the determinant does so, and furnishes a relation between the distances of $n+1$ points on a linear locus in n-dimensional space (Salmon's *Higher Algebra*, p. 22)*. If p_1 be the perpendicular from $(x_1, y_1, \dots)$ upon the linear locus through the other vertices, it follows that

$$\frac{p_1^2}{n^2} \cdot \frac{-1}{(-2)^{n-1}\{(n-1)!\}^2} \begin{vmatrix} 0, & 1, & 1, & \dots \\ 1, & 0, & (2.3)^2, & \dots \\ 1, & (2.3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{-1}{(-2)^n (n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

therefore

$$p_1^2 = -\tfrac{1}{2} \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 1, & (1.2)^2, & 0, & (2.3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \div \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ 1, & (2.3)^2, & 0, & (3.4)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

* This is proved directly, further on (Sec. II.)

Again,

$$V^2R^2 = -\frac{1}{(n!)^2}\begin{vmatrix} x_1, & y_1, & z_1, & \dots & R \\ x_2, & y_2, & z_2, & \dots & R \\ \dots & \dots & \dots & & \\ x_{n+1}, & y_{n+1}, & z_{n+1}, & \dots & R \end{vmatrix} \times \begin{vmatrix} x_1, & y_1, & z_1, & \dots & -R \\ x_2, & y_2, & z_2, & \dots & -R \\ \dots & \dots & \dots & & \\ x_{n+1}, & y_{n+1}, & z_{n+1}, & \dots & -R \end{vmatrix}$$

$$= -\frac{1}{(n!)^2}\begin{vmatrix} x_1^2+y_1^2 \dots -R^2, & x_1x_2+y_1y_2 \dots -R^2, & x_1x_3+y_1y_3 \dots -R^2, & \dots \\ x_1x_2+y_1y_2 \dots -R^2, & x_2^2+y_2^2 \dots -R^2, & x_2x_3+y_2y_3 \dots -R^2, & \dots \\ x_1x_3+y_1y_3 \dots -R^2, & x_2x_3+y_2y_3 \dots -R^2, & x_3^2+y_3^2 \dots -R^2, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= -\frac{1}{(n!)^2}\begin{vmatrix} 0, & -\frac{(1.2)^2}{2}, & -\frac{(1.3)^2}{2}, & \dots \\ -\frac{(1.2)^2}{2}, & 0, & -\frac{(2.3)^2}{2}, & \dots \\ -\frac{(1.3)^2}{2}, & -\frac{(2.3)^2}{2}, & 0, & \dots \end{vmatrix}$$

$$= \frac{-1}{(-2)^{n+1}(n!)^2}\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & 0, & (2.3)^2, & \dots \\ (1.3)^2, & (2.3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

and therefore

$$R^2 = -\tfrac{1}{2}\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, \dots \\ (1.2)^2, & 0, & (2.3)^2, \dots \\ (1.3)^2, & (2.3)^2, & 0, \dots \\ \dots & \dots & \dots \end{vmatrix} \div \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

If the $n+1$ points all lie on the same linear locus, $V=0$ (and since it has been assumed in the reduction that they all lie on a spherical locus), they are all situated upon a hyper-sphere in space of $n-1$ dimensions. Therefore

$$\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots & (1.n+2)^2 \\ (1.2)^2, & 0, & (2.3)^2, & \dots & (2.n+2)^2 \\ \dots & \dots & \dots & \dots & \dots \\ (1.n+2)^2, & (2.n+2)^2, & (3.n+2)^2, & \dots & 0, \end{vmatrix} = 0$$

is the condition that $n+2$ points in n-dimensional space should all lie on the same spheric.* This is a generalisation of a theorem of Professor Cayley's, given in Salmon's *Higher Algebra*, p. 21.

II. The identity of form between the formula for V^2 and the expression the vanishing of which expresses the identical relation between the lines joining $n+1$ points in space of $n-1$ dimensions, may be formally proved as follows.

Let (1`) be the projection of (1) on the opposite face of the simplicissimum, and $p_1$ the length of the perpendicular from the vertex (1) on that face; then $(1\,.\,r)^2=(1`\,.\,r)^2+p_1^2$;

therefore

$$V^2=\frac{-1}{(-2)^n\,(n\,!)^2}\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1\,.\,2)^2, & (1.3)^2, & \dots \\ 1, & (1\,.\,2)^2, & 0, & (2\,.\,3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$=\frac{-1}{(-2)^n\,(n\,!)^2}\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1`\,.\,2)^2+p_1^2, & (1`\,.\,3)^2+p_1^2, & \dots \\ 1, & (1`\,.\,2)^2+p_1^2, & 0, & (2\,.\,3)^2, & \dots \\ 1, & (1`\,.\,3)^2+p_1^2, & (2\,.\,3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$=\frac{-1}{(-2)^n\,(n\,!)^2}\times$$

$$\left\{\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1`\,.\,2)^2, & (1`\,.\,3)^2, & \dots \\ 1, & (1`\,.\,2)^2, & 0, & (2\,.\,3)^2, & \dots \\ 1, & (1`\,.\,3)^2, & (2\,.\,3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}+2p_1^2\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 0, & 0, & 1, & 1, & \dots \\ 1, & (1`\,.\,2)^2, & 0, & (2\,.\,3)^2, & \dots \\ 1, & (1`\,.\,3)^2, & (2\,.\,3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}\right.$$

$$\left.+p_1^4\begin{vmatrix} 0, & 0, & 1, & 1, & \dots \\ 0, & 0, & 1, & 1, & \dots \\ 1, & 1, & 0, & (2\,.\,3)^2, & \dots \\ 1, & 1, & (2\,.\,3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}\right\}$$

* For another proof, see further on (Sec. X.)

$$= \frac{-1}{(-2)^n (n!)^2} \times$$

$$\left\{ \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1`.2)^2, & (1`.3)^2, & \dots \\ 1, & (1`.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} + 2p_1^2 \begin{vmatrix} 0, & 1, & 0, & 0, & \dots \\ 0, & 0, & 1, & 1, & \dots \\ 1, & (1`.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \right\}$$

$$= \frac{-1}{(-2)^n (n!)^2} \times$$

$$\left\{ \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1`.2)^2, & (1`.3)^2, & \dots \\ 1, & (1'.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} - 2p_1^2 \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (2.3)^2, & (2.4)^2, & \dots \\ 1, & (2.3)^2, & 0, & (3.4)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \right\}$$

$$= \frac{-1}{(-2)^n (n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1`.2)^2, & (1`.3)^2, & \dots \\ 1, & (1`.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$+ \frac{2p_1^2}{(-2)^n (n!)^2} \times \frac{(-2)^{n-1} \{(n-1)!\}^2 V_1^2}{-1} *$$

$$= \frac{-1}{(-2)^n (n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1`.2)^2, & (1`.3)^2, & \dots \\ 1, & (1`.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} + \frac{p_1^2}{n^2} V_1^2,$$

and $V = \frac{p_1}{n} V_1$, and therefore the first determinant is equal to zero.

III. If R_1' be the radius of the spheric through all the vertices but one (1) of a simplicissimum, and through the centre of the circumscribed spheric, V_1' the content of the simplicissimum of which these points are the vertices, V_1 the content of the simplicissimum (of the $(n-1)^{\text{th}}$ order, of which (2), (3), &c. are the vertices,

$$V_1'^2 R_1'^2 = \frac{-1}{(-2)^{n+1} (n!)^2} \begin{vmatrix} 0, & R^2, & R^2, & R^2, & \dots \\ R^2, & 0, & (2.3)^2, & (2.4)^2, & \dots \\ R^2, & (2.3)^2, & 0, & (3.4)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

* Where V_1 is the content of the simplicissimum $(2.3 \dots n+1)$.

$$= \frac{-R^4}{(-2)^{n+1}(n!)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & (2.3)^2, & (2.4)^2, & \ldots \\ 1, & (2.3)^2, & 0, & (3.4)^2, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= -\frac{R^4}{(-2)^{n+1}(n!)^2} \times \frac{(-2)^{n-1}\{(n-1)!\}^2}{-1} V_1^2 = \frac{R^4 V_1^2}{4n^2};$$

therefore $V_1' = \frac{R^2 V_1}{2n R_1'}$, and $V = V_1' + V_2' + \ldots$

$$= \frac{R^2}{2n}\left\{\frac{V_1}{R_1'} + \frac{V_2}{R_2'} + \ldots\ldots\right\},$$

a formula which was set for space of three dimensions by Mr. Tebay (*Educational Times*, Feb., 1887, Qn. 8927).

This may also be proved otherwise. Let P_1 be the perpendicular from the centre of the circumscribed spheric upon the face opposite to the vertex (1), then

$$R^2 - P_1^2 = R_1'^2 - (R_1' - P_1)^2; \text{ therefore } R^2 = 2R_1' P_1,$$

and $$P_1 = \frac{R^2}{2R_1'}, \text{ but } V_1' = \frac{P_1}{n} V_1 = \frac{R^2 V_1}{2n R_1'},$$

and $$V = V_1' + V_2' \ldots\ldots$$

$$= \frac{R^2}{2n}\left\{\frac{V_1}{R_1'} + \frac{V_2}{R_2'} + \ldots\right\}.$$

If R_1 be the radius of the spheric about the simplicissimum $2.3 \ldots n+1$,

$$R_1^2 = R^2 - P_1^2 = 2R_1' P_1 - P_1^2$$

$$= \frac{R^2}{4R_1'^2}\{4R_1'^2 - R^2\},$$

$$R_1'^2 = \frac{R^4}{4(R^2 - R_1^2)} \text{ and } P_1^2 = R^2 - R_1^2.$$

IV. The formula for V is easily transformed into

$$\frac{1}{n!} \begin{vmatrix} x_1 - x_2, & y_1 - y_2, & \ldots \\ x_1 - x_3, & y_1 - y_3, & \ldots \\ \ldots & \ldots & \ldots \\ x_1 - x_{n+1}, & y_1 - y_{n+1}, & \ldots \end{vmatrix}$$

therefore

$$V^2 = \frac{1}{(n!)^2}\begin{vmatrix} (x_1-x_2)^2+(y_1-y_2)^2\ldots, & \left\{\begin{matrix}(x_1-x_2)(x_1-x_3)\\ +(y_1-y_2)(y_1-y_3)\ldots\end{matrix}\right\}, & \ldots \\ \left\{\begin{matrix}(x_1-x_2)(x_1-x_3)\\ +(y_1-y_2)(y_1-y_3)\ldots\end{matrix}\right\}, & (x_1-x_3)^2+(y_1-y_3)^2\ldots, & \ldots \\ \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots \\ (x_1-x_2)(x_1-x_{n+1})\ldots, & (x_1-x_3)(x_1-x_{n+1})\ldots, & \ldots \end{vmatrix}$$

= (taking the origin, as before, at the centre of the circumscribed spheric)

$$\frac{1}{2^n(n!)^2}\begin{vmatrix} 2\,(1.2)^2, & \left\{\begin{matrix}2R^2-2R^2+(1.2)^2-2R^2\\ +(1.3)^2+2R^2-(2.3)^2\end{matrix}\right\}, & \ldots \\ \left\{\begin{matrix}4R^2-4R^2+(1.2)^2\\ +(1.3)^2-(2.3)^2\end{matrix}\right\}, & 2\,(1.3)^2, & \ldots \\ \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots \\ \left\{\begin{matrix}4R^2-4R^2+(1.2)^2\\ +(1.n+1)^2-(2.n+1)^2\end{matrix}\right\}, & \ldots\quad\ldots & \ldots \end{vmatrix}$$

$$= \frac{(1.2)^2\,(1.3)^2\ldots(1.n+1)^2}{(n!)^2}\begin{vmatrix} 1, & \cos\alpha_{2.3}, & \cos\alpha_{2.4}, & \ldots \\ \cos\alpha_{2.3}, & 1, & \cos\alpha_{3.4}, & \ldots \\ \cos\alpha_{2.4}, & \cos\alpha_{3.4}, & 1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \cos\alpha_{2.n+1}, & \cos\alpha_{3.n+1}, & \cos\alpha_{4.n+1}, & \ldots \end{vmatrix}$$

if $\alpha_{r.s}$ denote the angle subtended at the vertex (1) by the edge $(r.s)$.

Hence the determinant

$$\begin{vmatrix} 1, & \cos\alpha_{2.3}, & \ldots \\ \cos\alpha_{2.3}, & 1, & \ldots \\ \ldots & \ldots & \ldots \\ \cos\alpha_{2.n+2}, & \cos\alpha_{3.n+2}, & \ldots \end{vmatrix}$$

vanishes, where all the $(n+2)$ vertices lie in a space of n dimensions [*i.e.*, on the same linear locus in space of $(n+1)$ dimensions].

It may be noticed that the determinant, in the expression for V^2 above, vanishes when the edges close up, and becomes unity when all the containing angles are right angles, and reduces to $\sin^2 A$ for space of two dimensions. It is, in fact, an analogue of that function, for

(1) In any simplicissimum the determinants of the different vertices

are proportional to the squares of the products of the edges of the opposite faces.

(2) If the determinants at the vertices of two simplicissima be equal, the simplicissima are proportional to the products of their coterminous edges.

(3) If the determinants corresponding to two vertices of a simplicissimum be equal, so are the products of the edges terminating at these vertices.

(4) For space of three dimensions it is the important function of the spherical triangle, determined by the solid angle at the vertex (1), which is usually denoted by $4n^2$, the vanishing of which shows that the triangle is evanescent, its angular points lying on the same great circle, and which therefore enters as a factor into the expressions for the sines of the angles of the triangle in terms of the sides, and into those for $\sin\frac{E}{2}$ and $\tan\frac{E}{4}$, and for $\cot R$ and $\cot R_1$ as well as those for $\tan r$ and $\tan r_1$, and it vanishes when either s, $s-a$, $s-b$, or $s-c$ is zero.

A similar transformation may be effected in the formula

$$V^2R^2 = \frac{-1}{(-2)^{n+1}(n!)^2}\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2 \ldots & (1.n+1)^2 \\ (1.2)^2, & 0, & (2.3)^2 \ldots & (2.n+1)^2 \\ \ldots & \ldots & \ldots & \ldots \\ (1.n+1)^2, & (2.n+1)^2, & \ldots & \ldots \end{vmatrix}$$

For, with the same notation for the angles between the edges at the vertex (1),

$$(r.s)^2 = (1.r)^2+(1.s)^2-2(1.r)(1.s)\cos\alpha_{r.s},$$

where r and s are any different numbers from 2 to $(n+1)$ inclusive.

Therefore $$V^2R^2 = \frac{-1}{(-2)^{n+1}(n!)^2}\times$$

$$\begin{vmatrix} 0, & (1.2)^2, & & (1.3)^2, & \ldots \\ (1.2)^2, & 0, & (1.2)^2+(1.3)^2-2(1.2)(1.3)\cos\alpha_{2.3} & & \ldots \\ (1.3)^2, & (1.2)^2+(1.3)^2-2(1.2)(1.3)\cos\alpha_{2.3}, & 0, & & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= -\frac{1}{(-2)^{n+1}(n!)^2} \times$$

$$\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots & (1.n+1)^2 \\ (1.2)^2, & -2(1.2)^2, & -2(1.2)(1.3)\cos a_{2.3}, & \dots & \dots \\ (1.3)^2, & -2(1.2)(1.3)\cos a_{2.3}, & -2(1.3)^2, & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= -\frac{2^{n-1}}{2^{n+1}(n!)^2} \times$$

$$\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & (1.2)^2, & (1.2)(1.3)\cos a_{2.3}, & \dots \\ (1.3)^2, & (1.2)(1.3)\cos a_{2.3}, & (1.3)^2, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= -\frac{(1.2)^2(1.3)^2 \dots (1.\overline{n+1})^2}{4(n!)^2} \begin{vmatrix} 0, & (1.2), & (1.3), & \dots \\ (1.2), & 1, & \cos a_{2.3}, & \dots \\ (1.3), & \cos a_{2.3}, & 1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

and therefore

$$R^2 = -\tfrac{1}{4} \begin{vmatrix} 0, & (1.2), & (1.3), & \dots & (1.n+1) \\ (1.2), & 1, & \cos a_{2.3}, & \dots & \cos a_{2.n+1} \\ (1.3), & \cos a_{2.3}, & 1, & \dots & \cos a_{3.n+1} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$\div \begin{vmatrix} 1, & \cos a_{2.3}, & \cos a_{2.4}, & \dots & \cos a_{2.n+1} \\ \cos a_{2.3}, & 1, & \cos a_{3.4}, & \dots & \cos a_{3.n+1} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

[Also, if V_1 be the content of the face opposite to the vertex 1,

$$V_1^2 = \frac{-1}{(-2)^{n-1}\{(n-1)!\}^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (2.3)^2, & (2.4)^2, & \dots \\ 1, & (2.3)^2, & 0, & (3.4)^2, & \dots \\ 1, & (2.4)^2, & (3.4)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{-1}{(-2)^{n-1}\{(n-1)!\}^2} \times$$

$$\begin{vmatrix} 0, & 1, & 1, & \dots \\ 1, & 0, & \left\{\begin{matrix}(1.2)^2+(1.3)^2 \\ -2\,(1.2)(1.3)\cos\alpha_{2.3}\end{matrix}\right\} & \dots \\ 1, & \left\{\begin{matrix}(1.2)^2+(1.3)^2 \\ -2\,(1.2)(1.3)\cos\alpha_{2.3}\end{matrix}\right\}, & 0, & \dots \\ 1, & \left\{\begin{matrix}(1.2)^2+(1.4)^2 \\ -2\,(1.2)(1.4)\cos\alpha_{2.4}\end{matrix}\right\}, & \left\{\begin{matrix}(1.3)^2+(1.4)^2 \\ -2\,(1.3)(1.4)\cos\alpha_{3.4}\end{matrix}\right\} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{-1}{(-2)^{n-1}\{(n-1)!\}^2} \times$$

$$\begin{vmatrix} 0, & 1, & 1, & \dots \\ 1, & -2\,(1.2)^2, & -2\,(1.2)(1.3)\cos\alpha_{2.3}, & \dots \\ 1, & -2\,(1.2)(1.3)\cos\alpha_{2.3}, & -2\,(1.3)^2, & \dots \\ 1, & -2\,(1.2)(1.4)\cos\alpha_{2.4}, & -2\,(1.3)(1.4)\cos\alpha_{3.4}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= -\frac{(1.2)^2(1.3)^2\dots(1.n+1)^2}{\{(n-1)!\}^2} \left[\begin{vmatrix} 0, & \frac{1}{(1.2)}, & \frac{1}{(1.3)}, & \dots \\ \frac{1}{(1.2)}, & 1, & \cos\alpha_{2.3}, & \dots \\ \frac{1}{(1.3)}, & \cos\alpha_{2.3}, & 1, & \dots \\ \frac{1}{(1.4)}, & \cos\alpha_{2.4}, & \cos\alpha_{3.4}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \right].*$$

V. In a paper I had the honour of reading here in December, 1883, I showed that in space of any dimensions, $(x, y, \dots)$ being the coordinates of any point, and

$$(x_1, y_1, \dots)(x_2, y_2, \dots)(x_3, y_3, \dots) \dots\dots (x_{n+1}, y_{p+1}, \dots)$$

those of $p+1$ known points, if $x, y, \dots$ were eliminated from the

* All passages enclosed in [ ] are additions to the paper as originally read.

equation to any locus by means of the equations

$$(\lambda+\mu+\ldots\tau)\,x = \lambda x_1+\mu x_2 \ldots\ldots \tau x_{p+1},$$
$$(\lambda+\mu+\ldots\tau)\,y = \lambda y_1+\mu y_2 \ldots\ldots \tau y_{p+1},$$
$$\&c. \qquad \&c. \qquad \&c.,$$

the resulting equation in $\lambda, \mu \ldots \tau$ would be the equation to the section of the given locus by the linear locus (in space of $p+1$ dimensions) determined by the $p+1$ given points, referred to what I should now call the simplicissimum whose vertices are at the $(p+1)$ points. The coordinates λ, μ, &c. are easily shown to be proportional to the contents of the simplicissima formed by replacing each vertex of the simplicissimum of reference in turn by the point $(\lambda, \mu \ldots)$.

If $\lambda+\mu+\ldots = V$, the coordinates will be the contents of these simplicissima, and I should propose to call them simplicissimum content coordinates. Areal and tetrahedral coordinates are instances of these.

When $p = n$, these same formulæ furnish a means of transforming to or from simplicissimum content coordinates.

VI. If the expression

$$\frac{1}{n!}\begin{vmatrix} \xi_1, & \eta_1 & \ldots\ldots & 1 \\ \xi_2, & \eta_2 & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ \xi_{n+1}, & \eta_{n+1} & \ldots\ldots & 1 \end{vmatrix}$$

for the content of the simplicissimum whose vertices are at

$$(\xi_1, \eta_1 \ldots), (\xi_2, \eta_2 \ldots) \ldots,$$

where $(x_1, y_1 \ldots)(x_2, y_2 \ldots)$, &c. are the vertices, and V the content of the simplicissimum of reference, be transformed by the equations

$$(\lambda+\mu \ldots \tau)\,x = \lambda x_1+\mu x_2+\ldots+\tau x_{n+1},$$
$$(\lambda+\mu \ldots \tau)\,y = \lambda y_1+\mu y_2+\ldots+\tau y_{n+1},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and $$(\lambda_1, \mu_1 \ldots), (\lambda_2, \mu_2 \ldots), \&c.$$

be the content coordinates of the vertices, the above determinant

$$= \frac{1}{n!\,V^{n+1}}\begin{vmatrix} \lambda_1 x_1+\mu_1 x_2+\ldots, & \lambda_1 y_1+\mu_1 y_2+\ldots, & \ldots \\ \lambda_2 x_1+\mu_2 x_2+\ldots, & \lambda_2 y_1+\mu_2 y_2+\ldots, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

$$= \frac{1}{n!} \cdot \frac{1}{V^{n+1}} \begin{vmatrix} \lambda_1, & \mu_1 & \dots \\ \lambda_2, & \mu_2 & \dots \\ \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1} & \dots \end{vmatrix} \times \begin{vmatrix} x_1, & y_1 & \dots & 1 \\ x_2, & y_2 & \dots & 1 \\ \dots & \dots & \dots & \\ x_{n+1}, & y_{n+1} & \dots & 1 \end{vmatrix}$$

$$= \frac{1}{V^n} \begin{vmatrix} \lambda_1, & \mu_1 & \dots \\ \lambda_2, & \mu_2 & \dots \\ \dots & \dots & \end{vmatrix},$$

an expression for the content of any simplicissimum in terms of the content coordinates of its vertices. The equation to the linear locus through $(\lambda_2, \mu_2 \dots)$, &c. is

$$\begin{vmatrix} \lambda, & \mu, & \dots \\ \lambda_2, & \mu_2, & \dots \\ \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1}, & \dots \end{vmatrix} = 0,$$

and therefore that to the parallel through $(\lambda_1, \mu_1, \dots)$ is

$$\begin{vmatrix} \lambda, & \mu & \dots \\ \lambda_2, & \mu_2 & \dots \\ \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1} & \dots \end{vmatrix} (\lambda_1 + \mu_1 \dots) = \begin{vmatrix} \lambda_1, & \mu_1 & \dots \\ \lambda_2, & \mu_2 & \dots \\ \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1} & \dots \end{vmatrix} (\lambda + \mu \dots),$$

and therefore for all points on this locus

$$\begin{vmatrix} \lambda, & \mu & \dots \\ \lambda_2, & \mu_2 & \dots \\ \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1} & \dots \end{vmatrix} = \begin{vmatrix} \lambda_1, & \mu_1 & \dots \\ \lambda_2, & \mu_2 & \dots \\ \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1} & \dots \end{vmatrix},$$

that is, simplicissima upon the same base and between the same parallels are equal.

VII. Let concurrent lines be drawn from the vertices of a simplicissimum, in space of n dimensions, through the point $(\lambda', \mu', \dots)$, to meet the opposite faces (simplicissima of the $\overline{n-1}^{\text{th}}$ order) in points whose content coordinates on the faces are $(\mu_1, \nu_1 \dots)$, $(\lambda_2, \nu_2 \dots)$, &c., then

$$\begin{array}{rrrrrrrrr} & \mu_1 & : & \nu_1 & : & \pi_1 \dots & :: & \mu' & : \nu' : \pi' \dots \\ \lambda_2 : & & & \nu_2 & : & \pi_2 \dots & :: & \lambda' & : \nu' : \pi' \dots \\ \lambda_3 : & \mu_3 & & & : & \pi_3 \dots & :: & \lambda' & : \mu' : \quad \pi' \dots \\ & \text{&c.} & & & & & & \text{&c.} & \end{array}$$

so that each term of $\begin{vmatrix} 0, & \mu_1, & \nu_1 \dots \\ \lambda_2, & 0, & \nu_2 \dots \\ \lambda_3, & \mu_3, & 0 \dots \\ \dots & \dots & \dots \end{vmatrix}$

has the same value, a result which includes the known results in Plane and Solid Geometry (*Educational Times*, Quest. 7980).

If P be the point $(\lambda', \mu' \dots)$ and A, B, &c. the vertices of the simplicissimum, and A', B', &c. the points where AP, BP, &c. meet the opposite faces,

$$\frac{PA'}{AA'} + \frac{PB'}{BB'} + \dots = \frac{\lambda'}{V} + \frac{\mu'}{V} + \dots = 1.$$

If $0, \mu_1', \nu_1' \dots$ be the n-ary content coordinates corresponding to the $(n-1)$-ary coordinates μ_1, ν_1, &c., and if p_1 be the perpendicular from the vertex 1 on $\lambda = 0$,

$$\mu_1 : \mu_1 + \nu_1 + \pi_1 \dots :: \mu' : \mu' + \nu' + \pi' \dots,$$

or
$$\mu_1 : \frac{nV}{p_1} :: \mu' : V - \lambda',$$

and
$$\mu_1' = \frac{p_1}{n}\mu_1 = \frac{V}{V-\lambda'}\mu', \text{ &c.}$$

Therefore the content of the simplicissimum whose vertices are at $(0, \mu_1', \nu_1' \dots)$, $(\lambda_2', 0, \mu_2' \dots)$, &c.

$$= \frac{V}{(V-\lambda')(V-\mu')\dots} \begin{vmatrix} 0, & \mu', & \nu' \dots \\ \lambda', & 0, & \nu' \dots \\ \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{V\lambda'.\mu'.\nu'}{(V-\lambda')(V-\mu')\dots} \begin{vmatrix} 0, & 1, & 1 \dots \\ 1, & 0, & 1 \dots \\ \dots & \dots & \end{vmatrix} = \frac{nV\lambda'.\mu'.\nu'}{(V-\lambda')(V-\mu')\dots}.$$

[Therefore the content of the simplicissimum $A'B'C'\dots$

$$= nV\frac{A'P.B'P\dots}{AP.BP\dots},$$

a formula given by Prof. Monck for the tetrahedron (*Educational Times*, Quest. 4208).

If a linear locus

$$l\lambda + m\mu + n\nu + \dots = 0$$

cut the edges of the simplicissimum of reference in P_{12}, P_{13}, P_{23}, &c., then

$$AP_{12} : BP_{12} :: -l : m,$$

&c. &c.

and there are $\frac{(n+1)n}{2}$ equations involving n independent ratios, and there are therefore $\frac{n(n-1)}{2}$ independent relations between the segments of the edges.

For a tetrahedron,

$$AP_{12} . BP_{23} . CP_{13} + BP_{12} . CP_{23} . AP_{13} = 0,$$

$$BP_{24} . CP_{23} . DP_{34} + BP_{23} . CP_{34} . DP_{24} = 0,$$

$$AP_{13} . CP_{34} . DP_{14} + AP_{14} . CP_{13} . DP_{34} = 0,$$

$$AP_{12} . BP_{24} . DP_{14} + AP_{14} . CP_{12} . DP_{24} = 0,$$

any three of which involve the fourth.]

VIII. If $(\lambda_1, \mu_1 \ldots), (\lambda_2, \mu_2 \ldots) \ldots (\lambda_{n+1}, \mu_{n+1} \ldots)$

be the vertices of a simplicissimum, this will be in perspective with the simplicissimum of reference if the equations

$$\frac{\mu'}{\mu_1} = \frac{\nu'}{\nu_1} = \ldots = k_1 \text{ say},$$

$$\frac{\lambda'}{\lambda_2} = \frac{\nu'}{\nu_2} = \ldots = k_2 \text{ say},$$

&c. &c.

$$\frac{\lambda'}{\lambda_{n+1}} = \frac{\mu'}{\mu_{n+1}} = \frac{\nu'}{\nu_{n+1}} = \ldots = k_{n+1} \text{ say},$$

are simultaneously true.

From these equations it follows that

$$\lambda_1 - V = \frac{1}{k_1}(\lambda' - V), \text{ &c.}$$

Now the equation of the face of the second simplicissimum corresponding to $\lambda = 0$, is

$$\begin{vmatrix} \lambda, & \mu, & \nu & \dots \\ \lambda_2, & \mu_2, & \nu_2 & \dots \\ \lambda_3, & \mu_3, & \nu_3 & \dots \\ \dots & \dots & \dots & \dots \\ \lambda_{n+1}, & \mu_{n+1}, & \nu_{n+1} & \dots \end{vmatrix} = 0,$$

and therefore where it meets $\lambda = 0$

$$\begin{vmatrix} 0, & \mu, & \nu & \dots \\ k_2\lambda_2, & k_2\mu_2, & k_2\nu_2 & \dots \\ k_3\lambda_3, & k_3\mu_3, & k_3\nu_3 & \dots \\ \dots & \dots & \dots & \dots \\ k_{n+1}\lambda_{n+1}, & k_{n+1}\mu_{n+1}, & k_{n+1}\nu_{n+1} & \dots \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} 0, & \mu, & \nu & \dots \\ \lambda', & \mu' + \sqrt{k_2 - 1}\, V, & \nu' & \dots \\ \lambda', & \mu', & \nu' + \sqrt{k_3 - 1}\, V & \dots \\ \dots & \dots & \dots & \dots \\ \lambda', & \mu', & \nu' & \dots \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \mu, & \nu, & \pi & \dots \\ k_2 - 1, & -(k_3 - 1), & 0 & \dots \\ 0, & k_3 - 1, & -(k_4 - 1) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0,$$

or

$$\frac{\mu}{k_2 - 1} + \frac{\nu}{k_3 - 1} + \dots = 0.$$

So that all the intersections of corresponding faces lie on the linear locus

$$\frac{\lambda}{k_1 - 1} + \frac{\mu}{k_2 - 1} + \frac{\nu}{k_3 - 1} + \dots = 0.$$

Again, the equations to the linear locus through the vertices $(\lambda_1, \mu_1 \dots)$, $(\lambda_2, \mu_2 \dots) \dots (\lambda_p, \mu_p \dots)$, are

$$\left\| \begin{matrix} \lambda, & \mu, & \nu & \dots \\ \lambda_1, & \mu_1, & \nu_1 & \dots \\ \lambda_2, & \mu_2, & \nu_2 & \dots \\ \dots & \dots & \dots & \\ \lambda_p, & \mu_p, & \nu_p & \dots \end{matrix} \right\| = 0,$$

$$\text{or} \quad \begin{Vmatrix} \lambda, & \mu, & \nu & \dots & \sigma, & \tau & \dots \\ \lambda'+\sqrt{k_1-1}\,V, & \mu', & \nu' & \dots & \sigma', & \tau' & \dots \\ \lambda', & \mu'+\sqrt{k_2-1}\,V, & \nu' & \dots & \sigma', & \tau' & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda', & \mu', & \nu & \dots & \sigma'+\sqrt{k_p-1}\,V, & \tau' & \dots \end{Vmatrix} = 0,$$

$$\text{or} \quad \begin{Vmatrix} \lambda, & \mu, & \nu & \dots & \sigma, & \tau & \dots \\ k_1-1, & -(k_2-1), & 0 & \dots & 0, & 0 & \dots \\ 0, & (k_2-1), & -(k_3-1) & \dots & 0, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda', & \mu', & \nu' & \dots & \sigma'+(k_p-1)\,V, & \tau & \dots \end{Vmatrix} = 0,$$

and where this meets the linear locus through the corresponding vertices of the tetrahedron of reference $\tau = 0$, &c., and therefore

$$\begin{vmatrix} \lambda, & \mu, & \nu & \dots & \sigma \\ k_1-1, & -(k_2-1), & 0 & \dots & 0 \\ 0, & (k_2-1), & -(k_3-1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & \dots & -(k_p-1) \end{vmatrix} = 0,$$

and therefore $$\frac{\lambda}{k_1-1}+\frac{\mu}{k_2-1}+\frac{\nu}{k_3-1}+\dots+\frac{\sigma}{k_p-1}=0,$$

and thus the linear loci through corresponding vertices intersect upon the same linear locus

$$\frac{\lambda}{k_1-1}+\frac{\mu}{k_2-1}+\dots=0.$$

In the special case where the vertices of the second simplicissimum are the feet of concurrent lines drawn from the vertices of the simplicissimum of reference, $(\lambda', \mu' \dots)$ being the point of concurrence as before, the coordinates of the vertices are

$$0, \frac{V\mu'}{V-\lambda'}, \frac{V\nu'}{V-\lambda'} \dots, \quad \frac{V\lambda'}{V-\mu'}, 0, \frac{V\nu'}{V-\mu'} \dots, \text{ &c.};$$

and therefore $$k_1=\frac{V-\lambda'}{V}, \quad k_2=\frac{V-\mu'}{V}, \text{ &c.},$$

and the equation to the locus obtained above becomes

$$\frac{\lambda}{\lambda'}+\frac{\mu}{\mu'}+\dots=0.$$

IX. If the transformation

$$(\lambda+\mu+\ldots+\tau)\,x = \lambda x_1+\mu x_2+\ldots+\tau x_{n+1},$$
$$(\lambda+\mu+\ldots+\tau)\,y = \lambda y_1+\mu y_2+\ldots+\tau y_{n+1},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

be applied to the equation

$$d^2 = (x-x')^2+(y-y')^2+\ldots,$$

$(\lambda, \mu, \nu \ldots)$ and $(\lambda', \mu' \ldots)$ being the content coordinates of $(x, y \ldots)$ and $(x', y' \ldots)$, since

$$\lambda+\mu+\ldots\,\tau = V = \lambda'+\mu'+\ldots\,\tau',$$

$$\begin{aligned}
V^2d^2 &= (\lambda+\mu\ldots+\tau)^2\,(x^2+y^2+z^2\ldots)\\
&\quad -2\,(\lambda+\mu\ldots+\tau)\,(\lambda'+\mu'\ldots+\tau')\,(xx'+yy'+\ldots)\\
&\quad +(\lambda'+\mu'+\ldots+\tau')^2\,(x'^2+y'^2+\ldots)\\
&= (\lambda x_1+\mu x_2+\nu x_3\ldots)^2+(\lambda y_1+\mu y_2+\ldots)^2+\ldots\\
&\quad -2\left\{(\lambda x_1+\mu x_2\ldots)(\lambda' x_1+\mu' x_2\ldots)+(\lambda y_1+\mu y_2\ldots)(\lambda' y_1+\mu' y_2\ldots)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.+\ldots\ldots\ldots\right\}\\
&\quad +(\lambda' x_1+\mu' x_2+\ldots)^2+(\lambda' y_1+\mu' y_2+\ldots)^2+\ldots\\
&= (\lambda-\lambda')^2\,(x_1^2+y_1^2+\ldots)+(\mu-\mu')^2\,(x_2^2+y_2^2+\ldots)+\ldots\\
&\quad +2\,(\lambda-\lambda')(\mu-\mu')\left\{x_1x_2+y_1y_2+\ldots\right\}-\&c.\\
&= \left\{(\lambda-\lambda')^2+(\mu-\mu')^2+\ldots\right\}R^2\\
&\quad +(\lambda-\lambda')(\mu-\mu')\left\{2R^2-(1.2)^2\right\}+(\mu-\mu')(\nu-\nu')\left\{2R^2-(2.3)^2\right\}\\
&\quad +\&c.
\end{aligned}$$

(if the origin of rectangular coordinates be at the centre of the circumscribed spheric)

$$\begin{aligned}
&= \left\{(\lambda+\mu\ldots)-(\lambda'+\mu'\ldots)\right\}^2R^2\\
&\quad -(\lambda-\lambda')(\mu-\mu')(1.2)^2-(\mu-\mu')(\nu-\nu')(2.3)^2-\&c.\\
&= -\left\{(\lambda-\lambda')(\mu-\mu')(1.2)^2+(\mu-\mu')(\nu-\nu')(2.3)^2+\&c.\right\},
\end{aligned}$$

which is, in fact, the result given by Professor Sylvester in the *Educational Times* (Quest. 8216), and which he has obtained in a more elegant manner (*Reprint*, Vol. XLV.). Professor Sylvester's coordinates are the same as those used here, if V be put $= 1$, for his x, y, &c. are the same as $\frac{\lambda}{V}$, $\frac{\mu}{V}$, &c.; or, writing

$$\lambda\mu\,(1.2)^2+\mu\nu\,(2.3)^2+\nu\lambda\,(1.3)^2+\ldots \equiv S$$

(as it is convenient to have a name for this function, which will frequently occur),

$$V^2d^2 = -S - S' + \lambda' \frac{dS}{d\lambda} + \mu' \frac{dS}{d\mu} + \dots ;$$

therefore the equation to a spherical locus, centre $(\lambda', \mu' \dots)$ and radius ρ, is

$$S - \frac{\lambda+\mu+\dots}{V} \left(\lambda \frac{dS'}{d\lambda'} + \mu \frac{dS'}{d\mu'} + \dots \right) + \frac{V^2\rho^2 + S'}{V^2} (\lambda+\mu+\dots)^2 = 0 ;*$$

and, since $\lambda+\mu \dots = 0$ is the equation to the linear locus at infinity, it appears that, in space of all dimensions, spherical loci all pass through a common spherical locus, in space of the next lower order, at infinity, and that every two have a common radical hyper-plane. And, since the theory of polars applies in space of n dimensions, it appears that the centre is the pole of infinity.

From the above equation,

$$S(\lambda'+\mu'+\dots)^2 - (\lambda+\mu \dots)(\lambda'+\mu'\dots) \left(\lambda \frac{dS'}{d\lambda'} + \mu \frac{dS'}{d\mu'} \dots \right) + S'(\lambda+\mu+\dots)^2 = 0$$

is the equation to the point spheric at $(\lambda', \mu' \dots)$.

X. In a similar way it is easily shown that

$$x^2+y^2+\dots = R^2,$$

the circumscribing spherical locus, is

$$\lambda\mu(1.2)^2 + \mu\nu(2.3)^2 + \nu\lambda(1.3)^2 + \dots = 0 \equiv S,$$

and hence that in general

$$(x^2+y^2+\dots) = \frac{1}{V^2} \{ \lambda^2 (x_1^2+y_1^2 \dots) + \mu^2 (x_2^2+y_2^2+\dots) + \dots + 2\lambda\mu (x_1x_2+y_1y_2 \dots) + 2\mu\nu (x_2x_3+y_2y_3 \dots) + \&c. \}$$

$$= \frac{1}{V^2} \{ (\lambda+\mu \dots)^2 R^2 - S \};$$

therefore

$$S = V^2 (R^2 - d^2),$$

where d is the distance of $\lambda, \mu, \nu \dots$ from the centre of the circumscribed spheric.

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* Since $S=0$ is, as will be shown (X.), the equation to the circumscribed spheric, it follows that the radical locus of the circumscribed and any other spheric is parallel to the polar of the centre of the latter with respect to the former.

If $(\lambda, \mu \ldots)$ be the centroid,

$$\lambda = \mu = \ldots = \frac{V}{n+1},$$

and
$$S = \frac{V^2}{(n+1)^2}\{(1.2)^2 + (2.3)^2 + \&c.\};$$

therefore
$$R^2 = \bar{d}^2 + \frac{1}{(n+1)^2}\{(1.2)^2 + (2.3)^2 + \ldots\},$$

as appeared before (Note to I.).

Since the centre is the pole of infinity, it is determined by the equations
$$\frac{dS}{d\lambda} = \frac{dS}{d\mu} = \frac{dS}{d\nu} \ldots\ldots,$$

If each of these $= k$,

$$\begin{vmatrix} V, & 1, & 1, & 1 & \ldots \\ k, & 0, & (1.2)^2, & (1.3)^2 & \ldots \\ k, & (1.2)^2, & 0, & (2.3)^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

and therefore

$$V \begin{vmatrix} 0, & (1.2)^2, & (1.3)^2 & \ldots \\ (1.2)^2, & 0, & (2.3)^2 & \ldots \\ (1.3)^2, & (2.3)^2, & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = -k \begin{vmatrix} 0, & 1, & 1, & 1 & \ldots \\ 1, & 0, & (1.2)^2, & (1.3)^2 & \ldots \\ 1, & (1.2)^2, & 0, & (2.3)^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

and, putting their values for the determinants,

$$k = 2R^2 . V,$$

and therefore

$$\lambda = \frac{1}{(-2)^n (n!)^2 V} \begin{vmatrix} 1, & (1.2)^2, & (1.3)^2 & \ldots & (1.n+1)^2 \\ 1, & 0, & (2.3)^2 & \ldots & (2.n+1)^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \&c.$$

Since
$$V^2 d^2 = -S - S' + \lambda' \frac{dS}{d\lambda} + \mu' \frac{dS}{d\mu} + \&c.,$$

where d is the distance between $(\lambda, \mu \ldots)$ an d $(', \mu' \ldots)$; if $(\lambda', \mu' \ldots)$ lie on the circumscribed spheric

$$V^2 d^2 = -S + \lambda' \frac{dS}{d\lambda} + \mu' \frac{dS}{d\mu} + \&c.,$$

and if $(\lambda', \mu' \ldots)$ be one of the vertices, say $\mu = \nu = \ldots = 0$,

$$V^2 d^2 = -S + V\frac{dS}{d\lambda} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and if $(\lambda, \mu \ldots)$ be the centre of the locus $S = 0$,

$$V\frac{dS}{d\lambda} = V^2R^2 + V^2R^2 \quad \text{and} \quad \frac{dS}{d\lambda} = 2VR^2,$$

as before.

[Hence, if AO, BO, &c. be produced to meet the opposite faces in A', B', &c., and the circum-spheric (again) in A'', B'', &c., and if $(\lambda, \mu \ldots)$ be O and $(0, \mu_1, \nu_1 \ldots)$ A', by (VII.),

$$\mu_1 = \frac{V}{V-\lambda}\mu, \quad \nu_1 = \frac{V}{V-\lambda}\nu, \text{ &c.},$$

therefore
$$V^2 . AA'^2 = -S_1 + V\frac{dS_1}{d\lambda_1}$$
$$= -\frac{V^2}{(V-\lambda)^2}\left(S - \lambda\frac{dS}{d\lambda}\right) + \frac{V^2}{V-\lambda}\frac{dS}{d\lambda},$$
$$= \frac{-V^2}{(V-\lambda)^2}(V^2R^2 - 2\lambda VR^2) + \frac{2V^3R^2}{V-\lambda},$$

therefore
$$AA'^2 = \frac{V^2R^2}{(V-\lambda)^2} \quad \text{and} \quad AA' = \frac{VR}{V-\lambda},$$

and
$$A'A'' = 2R - AA' = \frac{V-2\lambda}{V-\lambda}R.$$

Hence
$$\frac{1}{AA} + \frac{1}{BB'} + \ldots = \frac{n}{R},$$

and
$$\frac{A'A''}{AA'} + \frac{B'B''}{BB'} + \ldots = n-1.$$

{These properties are set for the triangle (*Educational Times*, Quest. 7660, Mr. Knowles, and 7335).}]

Again, if in (1) $S = 0$, so that $(\lambda, \mu \ldots)$ lies upon the circumscribed spheric, and d_1, d_2 ... denote the distances of this point from the respective vertices,

$$\frac{dS}{d\lambda} = Vd_1^2, \quad \frac{dS}{d\mu} = Vd_2^2, \text{ &c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and therefore

$$\lambda d_1^2+\mu d_2^2+\ldots=\frac{1}{V}\left\{\lambda\frac{dS}{d\lambda}+\mu\frac{dS}{d\mu}+\&c.\right\}=\frac{2}{V}S=0 \quad \ldots\ldots(3),$$

which proves the property expressed in the determinant form of the equation to the spheric, in orthogonal or other coordinates.

From the equations (2),

$$\begin{aligned} \mu\,(1.2)^2+\nu\,(1.3)^2+\ldots-Vd_1^2&=0,\\ \lambda\,(1.2)^2+\qquad\nu\,(2.3)^2+\ldots-Vd_2^2&=0,\\ \&c.\qquad\qquad\&c.& \end{aligned}$$

and from these equations and (3),

$$\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2 & \ldots & d_1^2 \\ (1.2)^2, & 0, & (2.3)^2 & \ldots & d_2^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ d_1^2, & d_2^2, & d_3^2 & \ldots & \ldots \end{vmatrix}=0.$$

If, then, $(1.n+2)$, $(2.n+2)$, &c. be written for d_1, d_2, &c.,

$$\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \ldots & (1.n+2)^2 \\ (1.2)^2, & 0, & (2.3)^2, & \ldots & (2.n+2)^2 \\ (1.3)^2, & (2.3)^2, & 0, & \ldots & (3.n+2)^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (1.n+2)^2, & (2.n+2)^2, & (3.n+2)^2, & \ldots & \ldots \end{vmatrix}=0$$

is, as was shown before (I.), the condition that $n+2$ points should lie on the same spheric in n-dimensional space, which may be transformed, as in (IV.), into

$$\begin{vmatrix} 0, & (1.2), & (1.3), & \ldots & (1.n+2) \\ (1.2), & 1, & \cos\alpha_{2.3}, & \ldots & \cos\alpha_{2.n+2} \\ (1.3), & \cos\alpha_{2.3}, & 1, & \ldots & \cos\alpha_{3.n+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}=0.$$

Since all spherical loci intersect in the same linear locus at infinity, and every two in another linear locus, the equation to a spheric may always be put in the form

$$S-(\lambda+\mu\ldots)\,(B_1\lambda+B_2\mu+\ldots)=0.$$

If this be compared with the general equation to a quadric locus,

$$A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu+\ldots=0.$$

It follows that

$$-kB_1 = A_{11}, \quad -kB_2 = A_{22}, \text{ \&c.},$$
$$k(1.2)^2 - k(B_1+B_2) = 2A_{12}, \text{ \&c.},$$

and therefore that $\dfrac{A_{11}+A_{22}-2A_{12}}{(1.2)^2} = \dfrac{A_{22}+A_{33}-2A_{23}}{(2.3)^2} = \text{\&c.},$

which give immediately the known conditions in Plane and Solid Geometry.

[If $$S-(a\lambda+b\mu+c\nu \ldots)(\lambda+\mu \ldots) = 0$$

be any spheric, and P_{12}, P'_{12}; P_{23}, P'_{23}, &c. be the points where it intersects the edges of the simplicissimum of reference,

$$a = AP_{12}.AP'_{12} = AP_{13}.AP'_{13} = \ldots,$$
$$b = BP_{12}.BP'_{12} = BP_{23}.BP'_{23} = \ldots,$$
$$\text{\&c.} \quad \text{\&c.} \quad \ldots \quad \text{\&c.}$$

for, if $\nu = \pi = \ldots = 0$,

$$a\lambda^2 + \{a+b-(1.2)^2\}\lambda\mu + b\mu^2 = 0,$$

and the values of $\lambda : \mu$ are $BP_{12} : AP_{12}$, and $BP'_{12} : AP'_{12}$;

therefore $$\frac{b}{a} = \frac{BP_{12}.BP'_{12}}{AP_{12}.AP'_{12}}, \quad \frac{(1.2)^2-(a+b)}{a} = \frac{BP_{12}}{AP_{12}} + \frac{BP'_{12}}{AP'_{12}};$$

therefore $$\left(1+\frac{BP_{12}}{AP_{12}}\right)\left(1+\frac{BP'_{12}}{AP'_{12}}\right) = \frac{(1.2)^2}{a},$$

and $$a = AP_{12}.AP'_{12} = \text{\&c.}]$$

XI. If, for shortness, A, B, &c. be written for $A_1\lambda+A_2\mu+$&c., $B_1\lambda+B_2\mu+$&c. &c., the equation

$$S-(\lambda+\mu+\nu \ldots)A-BC = 0$$

will represent a quadric locus through the spherical loci at infinity on $B = 0$ and $C = 0$, and consequently intersecting $B =$ constant and $C =$ constant in spherics in the $(n-1)$-dimensional spaces determined by those equations. Since a quadric in three-dimensional space can generally be reduced to this form, it follows such a quadric usually has two sets of parallel circular sections. If $B \equiv C$, the form is a special one, and the conditions that the surface should be one of revolution are obtained by equating the coefficients of the general form with those above. In the same way, the conditions that a quadric locus in higher space should be reducible to the form $S-(\lambda+\mu \ldots)A-BC = 0$ may be obtained.

Similarly, the equation

$$S-(\lambda+\mu\ldots)A-CD-DB-BC=0$$

represents a quadric locus, through the spherical loci at infinity, in the $(n-2)$-dimensional spaces determined by $B=0$, $C=0$; $C=0$, $D=0$; $D=0$, $B=0$; and consequently intersecting $B=$ const., $C=$ const.; $C=$ const., $D=$ const.; $D=$ const., $C=$ const., in spherical loci. As a quadric locus in space of four dimensions may in general be reduced to this form, it follows that such a locus has in general three sets of circular sections by parallel planes. The conditions that a quadric locus in higher space should be reducible to this form, may be obtained as above.

XII. The equations to the perpendicular to a given linear locus may now be obtained; for it is easily shown, by the use of orthogonal coordinates, that the line joining the centres of two spherical loci is at right angles to their radical locus. Let $(\lambda', \mu'\ldots)$, $(\lambda_1, \mu_1\ldots)$ be the centres of two such loci, then the equation to any parallel to the radical locus is

$$\lambda\left\{\frac{dS'}{d\lambda'}-\frac{dS_1}{d\lambda_1}\right\}+\mu\left\{\frac{dS'}{d\mu'}-\frac{dS_1}{d\mu_1}\right\}+\ldots\ldots$$
$$-k(\lambda+\mu+\ldots)=0,$$

and, if this be identical with $A_1\lambda+A_2\mu\ldots\ldots=0$,

$$\left.\begin{aligned}\frac{dS'}{d\lambda'}-\frac{dS_1}{d\lambda_1}-k&=A_1r\\ \frac{dS'}{d\mu'}-\frac{dS_1}{d\mu_1}-k&=A_2r\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$
$$\text{\&c.}\qquad\text{\&c.}\qquad\text{\&c.}$$

and

$$\left\|\begin{matrix}\frac{dS'}{d\lambda'}-\frac{dS_1}{d\lambda_1}, & \frac{dS'}{d\mu'}-\frac{dS_1}{d\mu_1}, & \ldots\\ A_1, & A_2, & \ldots\\ 1, & 1, & \ldots\end{matrix}\right\|=0;$$

and, since $(\lambda_1, \mu_1\ldots)$ is any point on the perpendicular from $(\lambda', \mu'\ldots)$ on $A_1\lambda+A_2\mu\ldots=0$, the equations to this are

$$\left\|\begin{matrix}\frac{dS'}{d\lambda'}(\lambda+\mu\ldots)-V\frac{dS}{d\lambda}, & \frac{dS'}{d\mu'}(\lambda+\mu\ldots)-V\frac{dS}{d\mu}, & \ldots\\ A_1, & A_2, & \ldots\\ 1, & 1, & \ldots\end{matrix}\right\|=0,$$

a form which is evidently unaltered when A_1+h, A_2+h, A_3+h, &c. are written for A_1, A_2, A_3, &c., as it ought to be.

If $(\lambda_1, \mu_1, \ldots)$ lie upon $A_1\lambda + A_2\mu + \ldots = 0$, the distance from $(\lambda', \mu', \ldots)$ to $(\lambda_1, \mu_1, \ldots)$ is the perpendicular from the former point upon the locus. Therefore

$$2V^2p^2 = -2S' - 2S_1 + 2\left(\lambda_1\frac{dS'}{d\lambda'} + \mu_1\frac{dS'}{d\mu'} \ldots\right),$$

but, from the equations (1),

$$0 = -2S_1 - kV + \left(\lambda_1\frac{dS'}{d\lambda'} + \mu_1\frac{dS'}{d\mu'} + \ldots\right);$$

therefore $$-2\,(V^2p^2 + S') + kV + \left(\lambda_1\frac{dS'}{d\lambda'} + \mu_1\frac{dS'}{d\mu'} + \ldots\right) = 0,$$

also $$A_1\lambda_1 + A_2\mu_1 + \ldots = 0,$$

$$-V + \lambda_1 + \mu_1 + \ldots = 0,$$

and $$-\frac{dS'}{d\lambda'} + k + A_1r + \mu_1\,(1.2)^2 + \nu_1\,(1.3)^2 + \ldots = 0,$$

$$-\frac{dS'}{d\mu'} + k + A_2r + \lambda_1\,(1.2)^2 + \nu_1\,(2.3)^2 + \ldots = 0,$$

&c., &c.,

$$\begin{vmatrix}
2\,(V^2p^2 + S'), & 0, & V, & \frac{dS'}{d\lambda'}, & \frac{dS'}{d\mu'}, & \frac{dS'}{d\nu'}, & \ldots \\
0, & 0, & 0, & A_1, & A_2, & A_3, & \ldots \\
V, & 0, & 0, & 1, & 1, & 1, & \ldots \\
\frac{dS'}{d\lambda'}, & A_1, & 1, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\
\frac{dS'}{d\mu'}, & A_2, & 1, & (1.2)^2, & 0, & (2.3)^2, & \ldots \\
\frac{dS'}{d\nu'}, & A_3, & 1, & (1.3)^2, & (2.3^2), & 0, & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0;$$

and, if L' stand for $A_1\lambda' + A_2\mu' + \ldots$, this reduces to

$$\begin{vmatrix}
2V^2p^2, & -L', & 0, & 0, & 0, & 0, & \ldots \\
-L', & 0, & 0, & A_1, & A_2, & A_3, & \ldots \\
0, & 0, & 0, & 1, & 1, & 1, & \ldots \\
0, & A_1, & 1, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\
0, & A_2, & 1, & (1.2)^2, & 0, & (2.3)^2, & \ldots \\
0, & A_3, & 1, & (1.3)^2, & (2.3)^2, & 0, & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0,$$

$$\text{or} \qquad 2V^2p^2 \begin{vmatrix} 0, & 0, & A_1, & A_2, & A_3, & \dots \\ 0, & 0, & 1, & 1, & 1, & \dots \\ A_1, & 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ A_2, & 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ & \dots & \dots & \dots & \dots & \end{vmatrix}$$

$$= L'^2 \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & (1.2)^2, & (1.3)^2 \\ 1, & (1.2)^2, & 0, & (2.3)^2 \\ & \dots & \dots & \dots \end{vmatrix} = L^2 \times \frac{(-2)^n}{-1} (n!)^2 V^2;$$

therefore

$$p^2 = (-2)^{n-1}(n!)^2 L'^2 \div \begin{vmatrix} 0, & 0, & A_1, & A_2, & A_3, & \dots \\ 0, & 0, & 1, & 1, & 1, & \dots \\ A_1, & 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ A_2, & 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ & \dots & \dots & \dots & \dots & \end{vmatrix} *$$

If $A_2 = A_3 = \&c. = 0$, so that the locus is $\lambda = 0$,

$$p^2 = (-2)^{n-1}(n!)^2 A_1^2 \lambda'^2 \div \begin{vmatrix} 0, & 0, & A_1, & 0, & 0, & \dots \\ 0, & 0, & 1, & 1, & 1, & \dots \\ A_1, & 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 0, & 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ & \dots & \dots & \dots & \dots & \end{vmatrix}$$

$$= (-2)^{n-1}(n!)^2 A_1^2 \lambda'^2 \div (-A_1^2) \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (2.3)^2, & (2.4)^2, & \dots \\ 1, & (2.3)^2, & 0, & (3.4)^2, & \dots \\ & \dots & \dots & \dots & \end{vmatrix}$$

$$= (-2)^{n-1} (n!)^2 A_1^2 \lambda'^2 \div (-2)^{n-1} \{(n-1)!\}^2 A_1^2 V_1^2$$

$$= \frac{n^2 \lambda'^2}{V_1^2}, \text{ as it should be.}$$

In this case the equations to the perpendicular are

$$\frac{dS'}{d\mu'}(\lambda+\mu+\dots) - (\lambda'+\mu'\dots)\frac{dS}{d\mu}$$

$$= \frac{dS'}{d\nu'}(\lambda+\mu+\dots) - (\lambda'+\mu'\dots)\frac{dS}{d\nu} = \&c.$$

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* This determinant is unaltered when A_1+h, A_2+h, &c. are written for A_1, A_2, &c., and is therefore unaltered for parallel loci.

If $(\lambda', \mu' \ldots)$ be the vertex $(V, 0, 0, \ldots)$, this becomes

$$(1\,.\,2)^2(\lambda+\mu+\ldots)-\frac{dS}{d\mu}$$

$$=(1\,.\,3)^2(\lambda+\mu+\ldots)-\frac{dS}{d\nu}$$

$$=\text{\&c.}\quad\text{\&c.}*$$

[If $l\lambda+m\mu+n\nu+\ldots=0$ be the equation to a linear locus, and p_1, p_2, &c. be the perpendiculars upon it from the vertices, it follows from the above that

$$p_1 : p_2 : \ldots :: l : m : \ldots,$$

and therefore, if such a locus touch a quadratic locus

$$A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu+\ldots=0,$$

$$\begin{vmatrix} 0, & p_1, & p_2, & p_3, & \ldots \\ p_1, & A_{11}, & A_{12}, & A_{13}, & \ldots \\ p_2, & A_{12}, & A_{22}, & A_{23}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix}=0.$$

Again, if, in the formula for the perpendicular upon the linear locus, λ be $=V$ and $\mu=\nu\ldots=0$, the perpendicular will be p_1, and if $l=p_1k$, $m=p_2k$, &c.,

$$2V^2p_1^2k^2\begin{vmatrix} 0, & 0, & p_1, & p_2, & \ldots \\ 0, & 0, & 1, & 1, & \ldots \\ p_1, & 1, & 0, & (1\,.\,2)^2, & \ldots \\ p_2, & 1, & (1\,.\,2)^2, & 0, & \ldots \\ & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$=k^2p_1^2V^2\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & (1\,.\,2)^2, & (1\,.\,3)^2, & \ldots \\ 1, & (1\,.\,2)^2, & 0, & (2\,.\,3)^2, & \ldots \\ & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

* It should have been stated that each of the equations to the line perpendicular to a given linear locus is the equation to a linear locus at right angles to it. And by the help of these the equations to any higher kind of linear locus at right angles to a given linear locus may be found.

or
$$\begin{vmatrix} 0, & 0, & p_1, & p_2, & p_3, & \dots \\ 0, & 0, & 1, & 1, & 1, & \dots \\ p_1, & 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ p_2, & 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ & \dots & \dots & \dots & \dots & \end{vmatrix} = (-2)^{n-1}(n!)^2 V^2;$$

which agrees with Professor Sylvester's results (Quests. 8042 & 8078, *Educational Times*).]

XIII. The spherical loci described about the $n+2$ simplicissima, each of which is bounded by $n+1$ out of $n+2$ given linear loci, all pass through the same point when n is even, but not, in general, when it is odd.

Let
$$\lambda = 0, \quad \mu = 0, \quad \nu = 0, \dots,$$
$$a\lambda + b\mu + c\nu \dots = 0 \equiv \eta,$$
be the $n+2$ linear loci.

The equation to the spheric about the simplicissimum of reference is
$$S = 0.$$
Let
$$S - (l\lambda + m\mu + n\nu \dots)(\lambda + \mu + \nu \dots) = 0$$
be that about the simplicissimum whose faces are $\mu = 0$, $\nu = 0 \dots \eta = 0$, therefore
$$l = 0, \quad m = \frac{-b(1.2)^2}{a-b}, \quad n = \frac{-c(1.3)^2}{a-c}, \text{ \&c.,}$$
and the equation is
$$S + \left\{ \frac{b(1.2)^2}{a-b}\mu + \frac{c(1.3)^2}{a-c}\nu + \dots \right\} \{\lambda + \mu + \dots\} = 0,$$
and where this cuts S,
$$\frac{b(1.2)^2}{a-b}\mu + \frac{c(1.3)^2}{a-c}\nu + \dots = 0,$$
and so for the other spherics.

Now
$$\frac{(1.2)^2}{a-b}b\mu + \frac{(1.3)^2}{a-c}c\nu + \dots = 0,$$
$$\frac{(1.2)^2}{b-a}a\lambda \qquad + \frac{(2.3)^2}{b-c}c\nu + \dots = 0,$$
$$\text{\&c.} \qquad \text{\&c.,}$$

all meet in a point, if

$$\begin{vmatrix} 0, & \frac{(1.2)^2}{a-b}, & \frac{(1.3)^2}{a-c}, & \dots \\ \frac{(1.2)^2}{b-a}, & 0, & \frac{(2.3)^2}{b-c}, & \dots \\ \frac{(1.3)^2}{c-a}, & \frac{(2.3)^2}{c-b}, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

vanish, which it always does if $n+1$ be odd, *i.e.*, if n be even. Hence the spherics all pass through a point if n be even (*in plano* this is Miquel's theorem). If n be odd, a condition must hold between a, b, &c., or their equivalents p_1, p_2, &c., the perpendiculars from the vertices on the locus $a\lambda+b\mu \dots = 0$. For space of three dimensions this condition is

$$\frac{(2.3)^2(1.4)^2}{(p_2-p_3)(p_1-p_4)} + \frac{(1.3)^2(2.4)^2}{(p_3-p_1)(p_2-p_4)} + \frac{(1.2)^2(3.4)^2}{(p_1-p_2)(p_3-p_4)} = 0.$$

XIV. Evidently, if a spheric touch each of the faces of the simplicissimum of reference, the centre $(\lambda', \mu' \dots)$ will be such that

$$\lambda' = \pm \frac{r}{n} V_1, \quad \mu' = \pm \frac{r}{n} V_2 \dots\dots,$$

the upper or lower sign being taken according as the spheric lies on the positive or negative side of the corresponding face, and therefore

$$\frac{r}{n} = \frac{\lambda'}{\pm V_1} = \frac{\mu'}{\pm V_2} = \&c.,$$

and there will be 2^n different spherics of the kind, the centres of which will lie upon what may be called the equidistant lines at each vertex as $\frac{\mu}{V_2} = \frac{\nu}{V_3} = \&c.$ (where the V's are scalars), which correspond to the bisectors of the angles of the triangle of reference in a plane.* If $(\mu_1, \nu_1 \dots)$, $(\lambda_2, \nu_2 \dots)$, &c. be the $(n-1)$-ary

* The equation $\frac{\mu}{V_2} = \frac{\nu}{V_3}$ is the equation to the linear locus, from which the perpendiculars upon $\mu = 0$ and $\nu = 0$ are equal, and so the equidistant lines from any vertex are those from which the perpendiculars upon all the faces which meet in that vertex are equal; and therefore, at the intersection of any two, those upon all the faces. They therefore pass by $n+1$ through each of the centres of the tangent spherics.

simplicissimum coordinates of the points where these meet the opposite faces,

$$\mu_1 : \nu_1 : \pi_1 \ldots :: \quad V_2 : V_3 : V_4 \ldots,$$

$$\lambda_2 : \quad \nu_2 : \pi_2 \ldots :: V_1 : \quad V_3 : V_4 \ldots,$$

$$\lambda_3 : \mu_3 : \nu_3 \quad \ldots :: V_1 : V_2 : V_3 \ldots,$$

$$\text{\&c.} \qquad \text{\&c.},$$

a series of relations analogous to those proved in Euclid (VI. 3 and A.).

Again $$\frac{r}{n} = \frac{V}{V_1 + V_2 \ldots},$$

and therefore $$r = \frac{nV}{V_1 + V_2 + \ldots}.$$

If r be the radius of the inscribed spheric, and r_1 that of the spheric for which λ' is negative, and μ', ν', &c. positive, then, p_1, p_2, &c. being the perpendiculars on the faces from the opposite vertices, since $nV = p_1 V_1 = p_2 V_2 = \text{\&c.}$,

$$\frac{1}{r} = \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \text{\&c.},$$

$$\frac{1}{r_1} = -\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \text{\&c.},$$

therefore $$\frac{1}{r} - \frac{1}{r_1} = \frac{2}{p_1}$$ (*Educational Times*, Qn. 7954),

an equation easily proved geometrically in space of two or three dimensions.

The equation to the inscribed spheric

$$\left(\text{Centre } \frac{nVV_1}{V_1 + V_2 + \ldots}, \quad \frac{nVV_2}{V_1 + V_2 + \ldots}, \quad \ldots \text{ radius } \frac{nV}{V_1 + V_2 + \ldots}\right),$$

and (with the appropriate changes in the signs of V_1, V_2, &c.) to the other spherics of this class, is

$$S - \frac{\lambda + \mu + \ldots}{V_1 + V_2 \ldots} \left\{ V_1 \frac{dS}{d\lambda} + V_2 \frac{dS}{d\mu} \ldots \right\}$$

$$+ \left(\frac{\lambda + \mu + \nu \ldots}{V_1 + V_2 \ldots}\right)^2 \left\{ n^2 V^2 + V_1 V_2 (1.2)^2 + V_2 V_3 (2.3)^2 \ldots \right\} = 0.$$

From which the equations to the inscribed and escribed circles of a triangle are easily obtained.

If D_1, D_2, &c. be the distances of the centre of the above spheric from the vertices, it is easy to show that

$$D_1^2 = -\{V_1V_2(1.2)^2 + V_2V_3(2.3)^2 + \ldots\}\frac{1}{(V_1+V_2+\ldots)^2}$$

$$+\{V_2(1.2)^2 + V_3(1.3)^2 + \ldots\}\frac{1}{V_1+V_2+\ldots},$$

&c. &c. &c.,

therefore $$V_1D_1^2 + V_2D_2^2 + \ldots = \frac{V_1V_2(1.2)^2 + V_2V_3(2.3)^2 + \ldots}{V_1+V_2+\ldots},$$

and $$D_1^2 - r^2 = -\{n^2V^2 + V_1V_2(1.2)^2 + V_2V_3(2.3)^2 + \ldots\}\frac{1}{(V_1+V_2+\ldots)^2}$$

$$+\frac{V_2(1.2)^2 + V_3(1.3)^2 + \ldots}{V_1+V_2+\ldots},$$

&c. &c.

If $(\mu_1, \nu_1 \ldots)$, $(\lambda_2, \nu_2 \ldots)$, &c. be the $(n-1)$-ary simplicissimum coordinates of the points of contact, it can be shown without difficulty that

$$\mu_1^2 = \lambda_2^2,\ \nu_1^2 = \lambda_3^2,\ \pi_1^2 = \lambda_4^2 \text{ &c.};\quad \nu_2^2 = \mu_3^2,\ \pi_2^2 = \mu_4^2, \text{ &c.}$$

&c. &c. &c.

XV.* [The centroid of a simplicissimum coincides with that of equal weights at its vertices, and that of its faces is the centre of the spheric inscribed in the figure whose vertices are the centroids of the faces, for, if the edges through $\mu = 0$, $\nu = 0$, ... be cut by a linear locus parallel to $\lambda = 0$, the content of the simplicissimum of the $n-1^{\text{th}}$ order whose vertices are at the intersections with these will be to the opposite face as $p^{n-1} : P_1^{n-1}$, where p and P_1 are the perpendiculars upon the new figure and upon the opposite face from $\mu = 0$, $\nu = 0$ ...; for, if the line perpendicular to the face $\lambda = 0$ from the opposite vertex be taken as axis of x and this point as origin,

$$\bar{x} = \frac{1}{V}\int_0^{P_1} \frac{V_1 p^{n-1}}{P_1^{n-1}} p\, dp$$

(where V_1 is the content of the face on $\lambda = 0$)

$$= \frac{V_1}{\frac{1}{n}V_1P_1^n}\int_0^{P_1} p^n\, dp = \frac{n}{n+1}P_1,$$

which proves the first statement.

The derived simplicissimum whose vertices are at the centroids of the faces is evidently similar to the original figure, and hence the centroid of all the faces but the one upon $\lambda = 0$ is the foot of the line equidistant from all the faces of the derived figure which meet upon $\lambda = 0$ (XIV.), and so for the others, and the second statement follows. This was set (for the tetrahedron) as a Question in the *Educational Times*, by Mr. J. J. Walker.]

XVI. The equation

$$ll'\lambda^2 + mm'\mu^2 + \&c. - (l'm + lm')\lambda\mu - \&c. = 0$$

represents a quadric locus which meets all the edges of the simplicissimum, and divides them in the ratios of $l : m$ and $l' : m'$, &c., respectively.

One special class of these, those for which $l' = l$, $m' = m$, &c., gives the figures which touch all the edges. In order that such a figure should be a spheric, it is necessary that

$$\frac{l^2 + m^2 + 2lm}{(1.2)^2} = \frac{m^2 + n^2 + 2mn}{(2.3)^2} = \&c.$$

$\frac{n(n+1)}{2} - 1$ homogeneous equations between $n+1$ quantities which lead to $\frac{(n+1)(n-2)}{2}$ conditions; so that such circles are always possible *in plano*, and the inscribed and escribed circles are easily found in this way.

But in higher space conditions arise; thus, for a sphere to touch the edges of a tetrahedron so that the inscribed circles of the triangular faces shall be sections of it, it is necessary that

$$(1.2) + (3.4) = (1.3) + (2.4) = (1.4) + (2.3),$$

while generally the inscribed circles of the triangles formed by any three vertices of a simplicissimum will be sections of the same spheric touching the edges if $d_1, d_2 \ldots d_{n+1}$, being a set of linear magnitudes, $(r.s) = d_r + d_s$ for all values of r and s (different) from 1 to $n+1$. And similar, but somewhat more complicated, conditions may be expressed for other cases. If in the general form $l' = m' = \ldots = 1$, the spheric, where such exists, will be cut by the plane through any three vertices in the nine-point circle of the triangle of which they are the angular points. The conditions for this are A_1, A_2, &c. being areal magnitudes $(r.s)^2 = A_r + A_s$ for all values of r and s (different) from 1 to $n+1$,

and the equation to the spheric is

$$A_1\lambda^2+A_2\mu^2+\ldots-(A_1+A_2)\,\lambda\mu-\&c.=0.*$$

In a similar manner the conditions may be arrived at that any set of circles, in the various triangles formed by the faces, should all lie upon one spheric.

In the general case the condition takes the form

$$\frac{(l+m)\,(l'+m')}{(1\,.\,2)^2}=\frac{(l+n)\,(l'+n')}{(1\,.\,3)^2}=\frac{(m+n)\,(m'+n')}{(2\,.\,3)^2}$$
$$=\ \&c.\qquad\&c.,$$

from which the ratios $l:m:n\ldots$ can be obtained whenever $l':m':n'\ldots$ are known.

XVII. As an example of the method of finding the equations to sections of loci by linear loci, mentioned above (V.), it may be well to prove that every such section of a spheric is a spheric.

Let the linear locus be that determined by the $p+1$ points

$$(\lambda_1,\ \mu_1\ldots),\quad(\lambda_2\ldots)\ \ldots\ (\lambda_{p+1}\ldots),$$

and let the coordinates of any point on it, referred to the simplicissimum of which these points are the vertices, be α, β, ... , then the

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* The above equation easily reduces to

$$(A_1+A_2)\,\lambda\mu+(A_2+A_3)\,\mu\nu+\ldots-\tfrac{1}{2}\,(\lambda+\mu+\ldots)\,(A_1\lambda+A_2\mu+\ldots)=0,$$

or
$$S-\tfrac{1}{2}\,(\lambda+\mu\ldots)\,(A_1\lambda+A_2\mu\ldots)=0.$$

Now, if there be a spheric self-conjugate with respect to the simplicissimum of reference, its equation

$$S-(\lambda+\mu\ldots)\,(B_1\lambda+B_2\mu\ldots)=0$$

must reduce to
$$B_1\lambda^2+B_2\mu^2+\ldots=0,$$

which involves the equations

$$B_1+B_2=(1\,.\,2)^2,\quad B_2+B_3=(2\,.\,3)^2,\quad\&c.,\quad\&c.,$$

and therefore
$$B_1=A_1\ \text{above},\quad B_2=A_2,\quad\&c.,$$

and the self-conjugate spheric and the nine-point spheric exist under the same conditions, and have a common radical locus with respect to $S=0$.

The conditions for the existence of these two spherics are also those that the perpendiculars, from the vertices on the opposite faces, should be concurrent. For the equations to the perpendicular (XII. p. 352), from $\mu=\nu=\ldots=0$ on $\lambda=0$, then reduce to
$$A_2\mu=A_3\nu=\ldots\&c.,$$

and the centre of perpendiculars is the point whose coordinates are as $\frac{1}{A_1}:\frac{1}{A_2}:\&c.$, and the locus of homology of the simplicissimum whose vertices are at the feet and the given simplicissimum is the above locus $A_1\lambda+A_2\mu\ldots=0$, by (VIII.).

equations of transformation are

$$(\alpha+\beta+\ldots+\theta)\,\lambda = \alpha\lambda_1+\beta\lambda_2\ldots+\theta\lambda_{p+1},$$

$$(\alpha+\beta+\ldots+\theta)\,\mu = \alpha\mu_1+\beta\mu_2\ldots\quad\theta\mu_{p+1}.$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

Also let $U \equiv S-(\lambda+\mu\ldots)(A_1\lambda_1+A_2\mu_2\ldots) = 0$;

and let $U_{1.1}$ denote the value of U when $(\lambda_1, \mu_1\ldots)$ are put for $(\lambda, \mu, \nu\ldots)$, and $U_{1.2}$ that of $\left\{\lambda\frac{dU_{2.2}}{d\lambda_2}+\mu\frac{dU_{2.2}}{d\mu_2}+\&c.\right\}$ under the same circumstances.

Then the equation to the section is

$$\alpha^2 U_{1.1}+\beta^2 U_{2.2}+\ldots+\alpha\beta U_{1.2}+\&c. = 0;$$

now
$$\begin{aligned} U_{1.1}+U_{2.3}-U_{1.2} &= \lambda_1\mu_1(1.2)^2+\mu_1\nu_1(2.3)^2+\ldots \\ &\quad-(A_1\lambda_1+A_2\mu_1\ldots)(\lambda_1+\mu_1\ldots) \\ &\quad+\lambda_2\mu_2(1.2)^2+\mu_2\nu_2(2.3)^2+\ldots \\ &\quad-(A_1\lambda_2+A_2\mu_2+\ldots)(\lambda_2+\mu_2\ldots) \\ &\quad-\{(\lambda_1\mu_2+\lambda_2\mu_1)(1.2)^2+(\mu_1\nu_2+\mu_2\nu_1)(2.3)^2+\ldots \\ &\quad+(A_1\lambda_1+A_2\mu_1\ldots)(\lambda_2+\mu_2\ldots) \\ &\quad+(A_1\lambda_2+A_2\mu_2\ldots)(\lambda_1+\mu_1\ldots)\} \\ &= (\lambda_1-\lambda_2)(\mu_1-\mu_2)(1.2)^2+(\mu_1-\mu_2)(\nu_1-\nu_2)(2.3)^2+\ldots \\ &= -V^2(1'.2')^2 \end{aligned}$$

if $(1'.2')$ represent the length of the line joining $(\lambda_1, \mu_1, \nu_1\ldots)$ and $\nu\,(\lambda_2, \mu_2\ldots)$; hence

$$\frac{U_{1.1}+U_{2.2}-U_{1.2}}{(1'.2')^2} = \frac{U_{2.2}+U_{3.3}-U_{2.3}}{(2'.3')^2} = \&c.,$$

and the section is a spheric.

It has not seemed advisable to lengthen the paper by giving the equations to the spheric through the centroids of the faces, that through the feet of the perpendiculars, &c., which are easily found in a manner similar to that by which the equation to the orthogonal spheric was obtained (p. 358, Note).

May 12th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. Alexander Anderson, B.A. Camb., Professor of Natural Philosophy, Queen's College, Galway, was elected a member.

The following papers were read :—

General Theory of Dupin's Space-extension of the Focal Properties of Conic Sections: J. Larmor.

Sur une propriété de la Sphère et son extension aux Surfaces quelconques: M. Maurice d'Ocagne.

On the Motion of Two Spaces in a Liquid, and allied Problems: A. B. Basset.

Second Note on Elliptic Transformation Annihilators: J. Griffiths.

The following presents were received :—

"American Journal of Mathematics," Vol. IX., 3; Baltimore.

"Educational Times," for May.

"Catalogue of the Library of the Royal Astronomical Society," 8vo; London, 1886.

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Part I.

"Proceedings of the Canadian Institute," March, 1887, No. 147, Vol. XXII.; Toronto, 1887.

"Annals of Mathematics," Vol. III., Nos. 1 and 2.

"Johns Hopkins University Circulars," Vol. VI., No. 57.

"Bulletin de la Société Mathématique de France," T. XV., No. 2.

"Bulletin des Sciences Mathématiques," T. XI., Mai, 1887.

"Beiblätter zu den Annalen der Physik und Chemie," B. XI., St. 4.

"Journal für die reine und angewandte Mathematik," Bd. C., H. 4; Bd. CI., H. I. and II.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., F. 5 and 6; Roma, 1887.

"Atti della Reale Accademia dei Lincei—Memorie," Vol. I.; Roma, 1885.

"Archives Néerlandaises des Sciences exactes et naturelles," T. XXI., Liv. 2, 3 and 4.

"Annali di Matematica," T. XIV., Fasc. 4; T. XV., Fasc. 1.

"Acta Mathematica," 9: 4.

"Bollettino delle Pubblicazioni Italiane, ricevute per diritto di Stampa," Nos. 31 and 32. Index to same for 1886—A to Ros.

"Zur Theorie der Binären Formen," by S. Gundelfinger, 4to pamphlet (from *Crelle*, Bd. 100, Heft. 4).

"Sur une certaine classe de Suites Récurrentes," by M. d'Ocagne (*Comptes Rendus*).

"Atti del Reale Istituto Veneto," T. III., Disp. 10; T. IV., Disp. 1 to 10 and Supplement; T. V., Disp. 1.

"Journal of the College of Science," Imperial University, Japan, Vol. I., Part I.; Tokio, 1886.

Sur une Propriété de la Sphère et son extension aux Surfaces quelconques. *Par* MAURICE D'OCAGNE.

[*Read May 12th*, 1887.]

Si le plan $Ax+By+Cz+1=0$

est tangent à la sphère $x^2+y^2+z^2=R^2$,

on a $$\frac{1}{\sqrt{A^2+B^2+C^2}}=R.$$

Par suite, la distance δ_i d'un point quelconque de l'espace $P_i\,(\alpha_i, \beta_i, \gamma_i)$ à ce plan tangent pourra s'écrire

$$\delta_i = R\,(A\alpha_i+B\beta_i+C\gamma_i+1).$$

Supposons alors qu'on prenne dans l'espace m points $P_1, P_2 \ldots P_m$ absolument quelconques. On aura, en appelant $\lambda_1, \lambda_2 \ldots \lambda_m$ des paramètres actuellement indéterminés

$$\Sigma\lambda_i\delta_i = R\,(A\Sigma\lambda_i\alpha_i+B\Sigma\lambda_i\beta_i+C\Sigma\lambda_i\gamma_i+\Sigma\lambda_i),$$

les signes Σ s'étendant aux valeurs 1, 2, 3, ..., m de l'indice i. Mais, lorsque m est au moins égal à 4, on peut toujours disposer des paramètres λ_i de façon que

$$\Sigma\lambda_i\alpha_i=0, \quad \Sigma\lambda_i\beta_i=0, \quad \Sigma\lambda_i\gamma_i=0,$$

ce qui revient à dire qu'on peut toujours considérer le centre de la sphère comme le barycentre des m points $P_1, P_2, \ldots, P_m$ affectés de certains coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$. Il vient alors, en représentant par k la somme des coefficients λ_i ainsi choisis,

$$\Sigma\lambda_i\delta_i = kR;$$

d'où ce théorème.

Il existe une relation linéaire entre les distances de m points quelconques de l'espace (*m étant au moins égal à* 4) *aux plans tangents à une sphère quelconque.*

On voit que le théorème est encore vrai pour le cas de $m=3$, lorsque le plan déterminé par les 3 points passe par le centre de la sphère, parce qu'alors le système d'équations

$$\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3\alpha_3=0,$$

$$\lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3=0,$$

$$\lambda_1\gamma_1+\lambda_2\gamma_2+\lambda_3\gamma_3=0,$$

est compatible.

Réciproquement :

Si les distances de m points fixes quelconques de l'espace à un plan mobile sont liées par une relation linéaire, ce plan enveloppe une sphère.

En effet, soit $$\Sigma\lambda_i\delta_i = K$$

la relation donnée. Prenons pour origine des coordonnées le barycentre O des points $P_1, P_2, \ldots, P_m$ respectivement affectés des coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$. Nous aurons alors

$$\Sigma\lambda_i\alpha_i = 0, \quad \Sigma\lambda_i\beta_i = 0, \quad \Sigma\lambda_i\gamma_i = 0,$$

et la relation précédente pourra s'écrire, en supposant que

$$Ax+By+Cz+1 = 0,$$

soit l'équation du plan mobile,

$$\frac{\Sigma\lambda_i}{\sqrt{A^2+B^2+C^2}} = K;$$

ce qui prouve que le plan mobile reste tangent à une sphère de centre O et de rayon $\frac{K}{\Sigma\lambda_i}$.

Soient $p_1, p_2, \ldots, p_m$ les pieds des perpendiculaires abaissées des points $P_1, P_2, \ldots, P_m$ sur l'un des plans tangents de la sphère O, et M le point de contact de ce plan tangent et de la sphère.

Puisque le point O est le barycentre des points $P_1, P_2, \ldots P_m$ affectés des coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$, il en est de même de la projection M du point O sur le plan tangent, relativement aux projections $p_1, p_2, \ldots, p_m$ des points $P_1, P_2, \ldots, P_m$ sur ce même plan.

Supposons maintenant que les distances $\delta_1, \delta_2, \ldots, \delta_m$ des m points fixes $P_1, P_2, \ldots, P_m$ à un plan π mobile soient liées par une relation absolument quelconque

$$F(\delta_1, \delta_2, \ldots, \delta_m) = 0.$$

Pour un déplacement infiniment petit du plan π, on a, par la formule de Taylor, en négligeant les infiniment petits du second ordre et des ordres supérieurs,

$$\frac{\partial F}{\partial\delta_1}d\delta_1 + \frac{\partial F}{\partial\delta_2}d\delta_2 + \ldots + \frac{\partial F}{\partial\delta_m}d\delta_m = 0.$$

C'est-à-dire que les déplacements infiniment petits que peut prendre le plan π à partir de la position considérée sont les mêmes que si les distances $\delta_1, \delta_2, \ldots, \delta_m$ étaient liées par la relation

$$\frac{\partial F}{\partial\delta_1}\delta_1 + \frac{\partial F}{\partial\delta_2}\delta_2 + \ldots + \frac{\partial F}{\partial\delta_m}\delta_m = \text{constante},$$

les dérivées partielles étant supposées avoir des valeurs fixes égales à celles qu'elles ont pour la position considérée du plan π. Le point de contact du plan π et de sa surface enveloppe est donc le même que son point de contact avec la sphère qu'il envelopperait en vertu de la relation précédente. Rapprochant ce résultat de la remarque qui a été faite plus haut, on obtient ce théorème, dont la ressemblance est manifeste avec un théorème célèbre de Tschirnhausen :

Si les distances $\delta_1, \delta_2, \ldots, \delta_m$ d'un plan mobile à m points fixes quelconques $P_1, P_2, \ldots, P_m$ de l'espace, sont liées par la relation

$$F(\delta_1, \delta_2, \ldots, \delta_m) = 0,$$

le plan mobile enveloppe une surface, et, pour chacune de ses positions, son point de contact avec cette surface est le barycentre des projections $p_1, p_2, \ldots, p_m$ des m points fixes sur ce plan, ces projections étant respectivement affectées des coefficients $\frac{\partial F}{\partial \delta_1}, \frac{\partial F}{\partial \delta_2}, \ldots, \frac{\partial F}{\partial \delta_m}$.

J'avais déjà obtenu ce théorème, par une voie toute différente, mais dans le cas seulement du plan* pour lequel je donnais, en outre, la détermination du rayon de courbure. La méthode précédente, en-dehors de l'avantage de s'appliquer à l'espace, a aussi celui d'être plus simple.

General Theory of Dupin's Space-Extension of the Focal Properties of Conic Sections. By J. Larmor.

[*Read May 12th*, 1887.]

The following considerations may have value in reducing to direct geometrical principles a remarkable theory whose treatment has usually been of an analytical and somewhat indirect character.

I. 1. The locus of a point in a plane, the sum or difference of whose distances from two fixed points in the plane is constant, is a conic section.

The locus of a point in space which possesses this property is a quadric of revolution. Any plane section of such a surface is a conic

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* Note I. de l'*Appendice* de ma brochure *Coordonnées parallèles et axiales* (Paris, Gauthier Villars, 1885).

section. It follows that there are for a conic section points outside its plane (the foci of the quadric) which possess this focal property. It is required to find their relations.

Let S_1, S_2 be the foci of the conic in its plane. The foci in space that are conjugate to S_1,—viz., that with S_1 give a focal relation,—belong to a certain locus which must include S_2. The locus will consist of two parts, say an elliptic part which corresponds to a constant sum of the distances, and a hyperbolic part which corresponds to a constant difference. Let $P_1, P_2, \ldots$ be points on the first part, and $Q_1, Q_2, \ldots$ points on the second part. We have then, if Π be any point on the conic,

$$S\Pi + P_1\Pi = \text{const.},$$

$$S\Pi + P_2\Pi = \text{const.},$$

$$S\Pi - Q_1\Pi = \text{const.},$$

$$S\Pi - Q_2\Pi = \text{const.}$$

Therefore

$$P_1\Pi - P_2\Pi = \text{const.},$$

$$P_1\Pi + Q_1\Pi = \text{const.}$$

Therefore any two points on the locus are conjugate foci; if they are on the same part of the locus their relation is hyperbolic, if on different parts it is elliptic.

It follows that the foci conjugate to *any* focus other than S also lie on the same locus.

Further, if P, Q are any two foci, and p_1, p_2 any two points on the conic,

$$Pp_1 \pm Qp_1 = \pm (Pp_2 \pm Qp_2),$$

the + or − sign being taken outside the bracket according as p_1, p_2 are on the same or different branches of the conic, and the other pairs of signs corresponding to each other; therefore

$$p_1P \mp p_2P = \mp (p_1Q \mp p_2Q),$$

$$= \text{constant}$$

for all points P, Q, ... on the locus.

Therefore the locus is a conic, of which any two points p_1, p_2 on the original conic are foci; the relation of these foci is hyperbolic if they lie on the same branch of the original conic, and elliptic if on different branches.

The relations of the locus-conic and the original one are therefore perfectly reciprocal. Each passes through the foci of the other. They clearly lie on perpendicular planes by symmetry, and the

vertices of each are therefore the ordinary foci of the other. This is Dupin's theorem.

There is, of course, an imaginary conic related to the ordinary imaginary foci on the minor axis, as the one here discussed is related to the ordinary real foci.

2. It is easy to place this in immediate relation to the elementary geometrical theory of a conic defined as a section of a right cone.

For the focus S of a section is the point of contact with its plane of a sphere inscribed in the cone. Therefore, if the line ΠO joining any point Π on the section to the vertex O of the cone touch this sphere in π, we have $\Pi\pi = \Pi S$. Therefore, as πO is of constant length,

$$O\Pi - S\Pi = \text{const.},$$

i.e., the vertex O of a right cone standing on the conic as base is a focus of the conic. Now, from consideration of the inscribed sphere, we have at once, as is well-known,

$$A_1O \pm A_2O = A_1S \pm A_2S = \text{const.},$$

A_1, A_2 being the vertices of the conic; so that the locus of O is a conic in a perpendicular plane with the vertices of the given one for foci and its foci for vertices.

The discrimination of the cases then follows as before.

[*Camb. Math. Tripos*, May 18, 1887; quoted in *Educational Times Reprint*, 1887.]

II. 3. Those considerations may now be extended to bicircular quartics without much difficulty.

The locus of a point in a plane whose distances r_1, r_2, r_3 from three fixed points in the plane are connected by a linear relation

$$ar_1 + br_2 + cr_3 = 0,$$

is a bicircular quartic with these points for foci. All quartic curves which have the circular points at infinity for nodes are bicircular, and may be defined in this way (Salmon's *Higher Plane Curves*, § 296), but one of these foci may be imaginary.

The locus of a point in space which possesses this property is a species of cyclide. Now all quartic surfaces which have the circle at infinity for a double line are cyclides. Any section of a cyclide, therefore, has the circular points at infinity in its plane for double points, *i.e.*, is a bicircular quartic. It follows that this bicircular quartic has also sets of foci (one set the foci of this surface) out of its plane; and we may proceed, as before, in the case of conics, to find their relations.

4. We notice that, as the inverse by reciprocal radii vectores of a plane bicircular quartic is a spherical curve which possesses the same focal property, and whose ordinary foci lie on the same sphere as itself, it follows that these more general spherical curves have also sets of foci outside their spherical surfaces.

Let us take P_1, P_2, P_3, a set of three foci of such a curve, and also Π_1, Π_2, Π_3, any three points on the curve. We have relations of the form

$$a_1 P_1\Pi_1 + a_2 P_2\Pi_1 + a_3 P_3\Pi_1 = 0,$$
$$a_1 P_1\Pi_2 + a_2 P_2\Pi_2 + a_3 P_3\Pi_2 = 0,$$
$$a_1 P_1\Pi_3 + a_2 P_2\Pi_3 + a_3 P_3\Pi_3 = 0.$$

Eliminating a_1, a_2, a_3, we obtain

$$\begin{vmatrix} \Pi_1 P_3, & \Pi_2 P_3, & \Pi_3 P_3 \\ \Pi_1 P_2, & \Pi_2 P_2, & \Pi_3 P_2 \\ \Pi_1 P_1, & \Pi_2 P_1, & \Pi_3 P_1 \end{vmatrix} = 0.*$$

If we take P_1, P_2 fixed points, this is a homogeneous linear relation connecting the distances of a variable point P_3 from the fixed points Π_1, Π_2, Π_3. Therefore all the foci conjugate to P_1, P_2 lie on a cyclide which has *any three points* Π_1, Π_2, Π_3 on the original curve for foci. But two such cyclides cannot coincide; therefore they intersect in a curve which is the locus.

Further, it is easy to see, as above, that any two foci, each of them conjugate to P_1 and P_2, form a pair which is conjugate to P_1 or P_2 separately; and that any three foci, each of them conjugate to P_1 and P_2, are conjugate to one another. Thus any three points on the locus of P_3 form a set of three conjugate foci; and the locus of P_3 includes the points P_1 and P_2.

The two curves are therefore reciprocally related, each being the locus of the foci of the other.

Moreover, as the locus is a quartic, and it cuts the plane of the original quartic in three real points, it must also cut it in a fourth, *i.e.*, the bicircular quartic has a fourth real focus in its plane.

It is not obvious from this method that the locus is itself a plane or spherical bicircular quartic; but, when we remember that the inverse of such a curve with respect to a focus is a Cartesian oval, for which the locus of foci must by symmetry lie in the perpendicular plane through its axis, which inverts again into a curve on a sphere,

* This equation is, I find, given by Darboux in his memoir, *Sur une classe remarquable* ..., § 19, to establish the reciprocal relation between $P_1P_2P_3$ and $\Pi_1\Pi_2\Pi_3$ but it does not lead him to the focal curve.

the result follows. The spheres on which the two conjugate quartics are situated intersect orthogonally.

The consideration that confines this argument to quartics on planes or spheres is the one at the beginning of Section II. The argument does not apply to space of more than three dimensions.

The curve which we have found as the locus of foci is a quartic on a sphere, and therefore the four ordinary foci in which it meets the plane or sphere of the original quartic lie on a circle. Any three of these four are conjugate to one another.

It is to be noted that, as for conics, there are other focal curves, three in all, which intersect the plane or sphere of the quartic, each in four foci, which are however all imaginary when the above-mentioned four are real; otherwise two of them may be real. (Darboux, l. c., or Casey.)

The quartics here found turn out to be focal curves of a system of confocal cyclides, just as the conics treated of in (I.) are the focal conics of a system of confocal quadrics. An investigation of their relations from that point of view, based upon the application of Ivory's theorem, will be found *Proc. Math. Soc.*, Vol. XVI., p. 189.

5. When one of the ordinary foci of the plane quartic is at an infinite distance, the circle on which the four foci lie becomes a right line, and the quartic is a Cartesian with equation of the form

$$ar_1 + br_2 = \text{const.},$$

where r_1, r_2 denote distances from two of the finite foci.

It is clear that, under the circumstances, the line at infinity is a part of the focal curve, and the remaining part is therefore a circular cubic. Conversely, the focal curve of a circular cubic is a Cartesian.

III. 6. These remarks have also an application to what is perhaps one of the most remarkable theories in Pure Geometry.

It has been shown by Chasles, in extension of Mr. M. Roberts' theorem, that if a thread has its extremities attached to two fixed points A, B on a quadric (i.), and is strained by a pencil-point P which is constrained to move on a confocal quadric (ii.), so that the parts of the thread near A and B lie along (i.), and the parts near P lie along (ii.), then the point P traces out a line of curvature on (ii.).

Applying the reasoning of § 1, it follows that *the whole surface* of the quadric (i.) may be similarly traced out by a thread whose extremities are attached to two points P, P' on a line of curvature of (ii.); which at first sight seems a result of much too sweeping a character. It is, however, true in the same sense as that the whole

surface of an ellipsoid may be traced out by a pencil constrained by a thread stretched over the surface between two opposite umbilics. The thread can be made to assume any azimuth on the surface, for there are an infinite number of geodesics which join two opposite umbilics, all necessarily of equal lengths.

7. In the case in question, the parameter (pD) of each geodesic portion of the thread on either surface has a constant value determined by the property that the tangent line to that geodesic touches a fixed confocal quadric, so that as one extremity of the thread is fixed at A on the first surface, the only result of moving the pencil P is to wind more or less thread along the geodesic through A without disturbing the part already in contact with the surface. Instead of A we may therefore fix any other point A' on the geodesic; so that the restriction that A, B lie on the same line of curvature is withdrawn.

Again, the two geodesic arcs of PA, PB on the second quadric have the same parameter, which shows that either

(a) They are equally inclined to the lines of curvature through P, which leads to Chasles' theorem; or

(b) They are in one straight line, in which case through any point P can be drawn a continuous line geodesically connecting A and B; and since each of these geodesic connecting lines is the shortest distance between A and B, subject to the constraint of the surfaces, they are all of equal length, and the explanation suggested is verified.

We have therefore the two results—

(a) A pencil P guided by a string stretched slackly between A and B traces out a line of curvature on (ii.); while

(b) A string stretched tightly between A and B can be made to slip over the surfaces with perfect freedom.

8. To obtain a clearer representation, we bear in mind that on an ellipsoid a line of curvature of given pD consists of two closed curves on opposite faces of the surface, and that a geodesic of that pD may be imagined as a thread wound on the surface as on a reel, so as to touch each branch in every revolution. Usually the thread will be endless, and will in a sense form the whole geodesic system of that parameter; in certain cases it will be re-entrant.

Whatever be the points A and B on (i.), the portion of the thread on (ii.) will be an arc of that geodesic system which touches the curve of intersection of (i.) and (ii.).

Let us then imagine a thread wrapped round and round the sur-

face (ii.), so as to touch alternately the two ovals of this curve of intersection. Taking any portion of it, the tangent lines along it touch the surface (i.), and, when produced geodesically along it, form the corresponding arcs of the geodesic system of (i.), *i.e.*, those arcs which run in the same direction, in the sense that we can proceed by continuous change from the one to the other through the intermediate arcs.

If, therefore, we take any two points A and B on (i.), the corresponding geodesic arcs drawn from them, and carried geodesically on to (ii.), finally run into one another and are continuous.

Again, if we take any point on (ii.) and follow the two geodesic arcs of the system which cross each other there, they will run on to (i.) so as to be non-corresponding geodesic arcs on it.

If, therefore, we take any two points A and B on (i.), the non-corresponding geodesic arcs drawn from them and carried geodesically on to (ii.) will meet in a point P on (ii.), such that the sum or difference of AP and BP is constant, according to the relative position of A and B considered fixed, in a way that is most easily illustrated by the analogy of Roberts' theorem for a single quadric.

10. If the quadric (i.) is taken to be a focal conic, and the points A and B are taken on its boundary, all lines through A and B are tangential to (i.), so that the condition of geodesic contact with (i.) imposes no restriction.

Thus taking, for definiteness, (ii.) to be an ellipsoid, we see that a thread drawn tight over it between two points A and B on opposite quadrants of the focal hyperbola is free to slip all over it; and also that the locus of the space-foci, in the geodesic sense, of the lines of curvature is the focal hyperbola, which, of course, contains the umbilics.

On the Motion of Two Spheres in a Liquid, and Allied Problems.

By A. B. Basset, M.A.

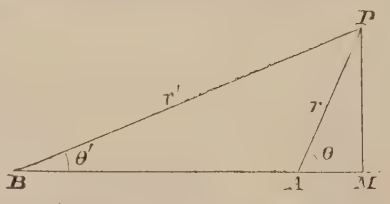
[*Read May 12th*, 1887.]

1. When two spheres are moving in an infinite liquid along the line joining their centres, the coefficients of the velocities in the expression for the kinetic energy, have been completely determined by the method of images, in the form of infinite series, by Mr. W. M.

Hicks*; but when the spheres are moving perpendicularly to the line joining their centres, the successive images become so very complicated that Mr. Hicks has been compelled to abandon the attempt of obtaining the complete solution, and to resort to an approximate one.

In the present paper I propose to explain a different method, by means of which approximate values of the coefficients may be obtained in a series of powers of the inverse distance of the centres of the two spheres, and the calculation is carried as far as the inverse twelfth power. I have also pointed out that the same method may be applied to obtain an approximate solution of the problem of two electrified spheres, which are placed in a field of force whose potential is given.

2. It will first be necessary to establish the following proposition.



In the figure, let $PM = \rho$, $AM = z$, $BM = z'$, $AB = c$, $\cos\theta = \mu$, $\cos\theta' = \mu'$; also let $P_n^m(\mu)$ be an associated function of degree n and order m, whose origin is A, and axis is AM; and let $P_n'^m(\mu')$ denote a similar function having the same axis and whose origin is B. Then we shall prove that, when $r < c$,

$$\frac{P_n'^m}{r'^{n+1}} = \frac{r^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m - \frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P_{m+1}^m + \ldots + \frac{(-)^s(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P_{m+s}^m + \ldots\right] \ldots (1),$$

and when $r' < c$,

$$\frac{(-)^{n-m}P_n^m}{r^{n+1}} = \frac{r'^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m'^m + \frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P_{m+1}'^m + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P_{m+s}'^m + \ldots\right] \ldots (2).$$

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* *Phil. Trans.*, 1880.

[A complete account of the history of this problem, together with a description of the papers by Bjerknes and others, will be found in Mr. W. M. Hicks' Report on Hydrodynamics to the British Association, 1882, Part II., pages 13 to 16. Since the publication of this report, the following papers have appeared:—

K. Pearson, "On the Motion of Spherical and Ellipsoidal Bodies in Fluid Media." *Quarterly Journal*, Vol. xx., p. 60.

A. H. Leahy, "On the Pulsations of Spheres in an Elastic Medium," *Camb. Trans.*, Vol. xiv., p. 45.

R. A. Herman, "On the Motion of Two Spheres in a Fluid, and Allied Problems," *Quarterly Journal*, Vol. xxii., p. 204 (June, 1887).

In the last paper the approximation is carried as far as the *fifteenth* power of the inverse distance between the centres of the spheres (see p. 229), and the results agree with those given in the present paper. (*June 10th.*)]

It is known that P_n^m can be expressed in either of the forms

$$M(1-\mu^2)^{\frac{1}{2}m}\int_0^\pi \left\{\mu+\sqrt{\mu^2-1}\cos\phi\right\}^{n-m}\sin^{2m}\phi\,d\phi,$$

or

$$M(1-\mu^2)^{\frac{1}{2}m}\int_0^\pi \frac{\sin^{2m}\phi\,d\phi}{\left\{\mu+\sqrt{\mu^2-1}\cos\phi\right\}^{n+m+1}},$$

where

$$M=\frac{(n+m)!}{(n-m)!\,1.3\ldots(2m-1)\,\pi}.$$

Therefore

$$\frac{P_n^m(\mu')}{r'^{n+1}}=M\rho^m\int_0^\pi\frac{\sin^{2m}\phi\,d\phi}{(z'+\rho\cos\phi)^{n+m+1}}$$

$$=M\rho^m\int_0^\pi\frac{\sin^{2m}\phi\,d\phi}{\left[c+r\left\{\mu+\sqrt{\mu^2-1}\cos\phi\right\}\right]^{n+m+1}};$$

whence, if $\lambda=\mu+\sqrt{\mu^2-1}\cos\phi$, and $r<c$,

$$\frac{P_n^m(\mu')}{r'^{n+1}}=\frac{M\rho^m}{c^{n+m+1}}\int_0^\pi\left\{1-(n+m+1)\frac{r\lambda}{c}+\frac{(n+m+1)(n+m+2)}{2!}\left(\frac{r\lambda}{c}\right)^2\ldots\right.$$

$$\left.\ldots+\frac{(-)^s(n+m+1)\ldots(n+m+s)}{s!}\left(\frac{r\lambda}{c}\right)^s+\ldots\right\}\sin^{2m}\phi\,d\phi;$$

whence, by the first form of P_n^m, we obtain

$$\frac{P_n^m}{r'^{n+1}}=\frac{r^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m-\frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P_{m+1}^m+\ldots\right.$$

$$\left.\ldots+\frac{(-)^s(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P_{m+s}^m+\ldots\right].$$

In order to obtain the second equation, let us change θ and θ' into their supplements; then, since

$$P_n^m\left\{\cos(\pi-\theta)\right\}=(-)^{n-m}P_n^m(\cos\theta),$$

we obtain

$$\frac{(-)^{n-m}P_n^m(\mu)}{r'^{n+1}}=\frac{r'^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m+\frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P_{m+1}^m\right.$$

$$\left.\ldots+\frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P_{m+s}^m+\ldots\right].$$

The corresponding formulæ when $r>c$ could be easily obtained, but they are not required for the present investigation.

3. Let A and B be the centres of the two moving spheres, a and b their radii, v_1, v_2 their parallel component velocities perpendicular to

AB. Also let ϕ_1 be the velocity potential of the liquid when A is moving with velocity v_1, whilst B is kept at rest, and let ϕ_2 be the velocity potential when B is in motion and A is fixed. Then if ϕ be the velocity potential of the whole motion,

$$\phi = \phi_1 + \phi_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The problem is therefore reduced to the determination of ϕ_1, for when this is known, ϕ_2 can be written down by symmetry.

Let χ be the angle which a plane through AB and any point P makes with the plane through AB which contains the directions of motion of A and B; also let Q_n, Q'_n be written for P^1_n and P'^1_n. Then, in the neighbourhood of A, ϕ_1 must be expressible in the form of the series

$$\phi_1 = \left\{ -\frac{v_1 a^3 Q_1}{2r^2} + A_1 \left(r + \frac{a^3}{2r^2} \right) Q_1 + A_2 \left(r^2 + \frac{2a^5}{3r^3} \right) + \ldots \right\} \cos\chi \quad \ldots (4),$$

for this value of ϕ_1 satisfies the surface condition

$$\left(\frac{d\phi_1}{dr} \right)_a = v_1 \sin\theta \cos\chi.$$

In the neighbourhood of B, ϕ_1 must be expressible in the form

$$\phi_1 = \left\{ B_1 \left(r' + \frac{b^3}{2r'^2} \right) Q'_1 + B_2 \left(r'^2 + \frac{2b^5}{3r'^3} \right) Q'_2 + \ldots \right\} \cos\chi \quad \ldots\ldots (5),$$

for the value of ϕ_1 satisfies the surface condition

$$\left(\frac{d\phi_1}{dr'} \right)_b = 0.$$

The series consisting of powers of r^{-1} and r'^{-1} are convergent at all points outside the two spheres, but the series consisting of powers of r and r' will be divergent if r and r' be sufficiently great; but we shall only require these latter series in the neighbourhood of the two spheres where they are convergent.

The kinetic energy consists of a series of terms of the form

$$\begin{aligned} \int_0^{2\pi} d\chi \int_0^{\pi} Q_n v_1 a^2 \cos^2\chi \sin^2\theta \, d\theta &= \pi a^2 v_1 \int_{-1}^{1} (1-\mu^2) \frac{dP_n}{d\mu} d\mu \\ &= 2\pi a^2 v_1 \int_{-1}^{1} \mu P_n \, d\mu \\ &= \frac{4\pi a^2 v_1}{3} \ (n=1) \quad \ldots\ldots\ldots\ldots (6) \\ &= 0 \quad (n \text{ any other value}). \end{aligned}$$

Hence the terms involving Q_2, Q_3, &c. contribute nothing to the energy, and we may therefore, in writing down the final value of ϕ_1, reject all terms except those involving Q_1 or Q_1'.

Dropping the factor $\cos\chi$ for the present, we should have, if B were absent,

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2}.$$

Putting $m=1$, $n=1$ in (2), the value of this near B is

$$\phi_1 = -\frac{v_1 a^3}{2c^3}\left(r'Q_1' + \frac{r'^2 Q_2'}{c} + \frac{r'^3 Q_3'}{c^3} + \frac{r'^4 Q_4'}{c^4} + \ldots\right).$$

From (5) it follows that, in order to make the velocity at B vanish, we must add the series

$$-\frac{v_1 a^3}{2c^3}\left(\frac{b^3 Q_1'}{2r'^2} + \frac{2b^5 Q_2'}{3cr'^3} + \frac{3b^7 Q_3'}{4c^2 r'^4} + \frac{4b^9 Q_4'}{5c^3 r'^5} + \ldots\right).$$

Transforming this latter series back again to A by (1), and retaining the important terms only, the value of ϕ_1 near A becomes

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2} - \frac{v_1 a^3 b^3}{c^6}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{4b^6}{c^6}\right) Qr + \frac{v_1 a^3 b^3}{4c^7} Q_2 r^2.$$

In order to satisfy the surface conditions at A, add the terms

$$-\frac{v_1 a^3 b^3}{c^6}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{4b^6}{c^6}\right)\frac{Q_1 a^3}{2r^2} + \frac{v_1 a^3 b^3 Q_2}{6c^7 r^3}.$$

Neglecting powers of c^{-1} higher than the twelfth, the value of these added terms near B is

$$-\frac{v_1 a^6 b^3}{2c^9}\left(\frac{1}{4} + \frac{b^2}{c^2}\right) Q_1' r' - \frac{v_1 a^8 b^3}{2c^{11}} Q_1' r'.$$

Adding the terms $$-\frac{v_1 a^6 b^3}{4c^9}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{a^2}{c^2}\right)\frac{Q_1' b^3}{r'^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

omitting Q_2', &c., and restoring $\cos\chi$, the value of the velocity potential near B becomes

$$\phi_1 = -\frac{v_1 a^3}{2c^3}\left\{1 + \frac{a^3 b^3}{4c^6} + \frac{a^3 b^3 (a^2+b^2)}{c^8}\right\} r' + \frac{b^3}{2r'^2}\Big) Q_1' \cos\chi \ \ldots (8).$$

The first term of (7) on transformation becomes

$$-v_1 a^6 b^6 Q_1 r/16c^{12},$$

whence the value of ϕ_1 near A is

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2}\cos\chi - \frac{v_1 a^3 b^3}{c^6}\left\{\frac{1}{4}+\frac{b^2}{c^2}+\frac{9b^4}{4c^4}+\frac{(a^3+64b^3)\,b^3}{16c^6}\right\} \times\left(r+\frac{a^3}{2r^2}\right)Q_1\cos\chi \quad \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The values of ϕ_2 at A and B can be written down by symmetry; whence, if T be the kinetic energy of the system

$$2T = A'v_1^2+2B'v_1v_2+C'v_2^2,$$

where

$$\left.\begin{aligned} A' &= m_1-\iint\phi_1\frac{d\phi_1}{dn}\,dS_1 \\ &= m_1+\tfrac{1}{2}M_1\left[1+\frac{3a^3b^3}{c^6}\left\{\frac{1}{4}+\frac{b^2}{c^2}+\frac{9b^4}{4c^4}+\frac{b^3\,(a^3+64b^3)}{16c^6}\right\}\right] \\ C' &= m_2+\tfrac{1}{2}M_2\left[1+\frac{3a^3b^3}{c^6}\left\{\frac{1}{4}+\frac{a^2}{c^2}+\frac{9a^4}{4c^4}+\frac{a^3\,(b^3+64a^3)}{16c^6}\right\}\right] \\ B' &= -\rho\iint\phi_1\frac{d\phi_2}{dn}\,dS_2 \\ &= \frac{\pi\rho a^3b^3}{c^3}\left\{1+\frac{a^3b^3}{4c^6}+\frac{a^3b^3\,(a^2+b^2)}{c^8}\right\} \end{aligned}\right\} \ldots(10),$$

where m_1, m_2 are the masses of the spheres A and B; M_1, M_2 those of the liquid displaced by them, and ρ is the density of the liquid.*

4. When the motion takes place along the line of centres, the energy could be calculated in a similar manner; but in this case it is better to employ the method of images. By means of the formulæ given on p. 463 of Mr. Hicks' paper, it can easily be shown that, if the kinetic energy be $(Au_1^2-2Bu_1u_2+Cu_2^2)/2$, then

$$\left.\begin{aligned} A &= m_1+\tfrac{1}{2}M_1\left\{1+\frac{3a^3b^3}{c^6}\left(1+\frac{3b^2}{c^2}+\frac{6b^4}{c^4}+\frac{11b^6}{c^6}\right)\right\} \\ B &= \frac{2\pi\rho a^3b^3}{c^3}\left\{1+\frac{a^3b^3}{c^6}+\frac{3a^3b^3\,(a^2+b^2)}{c^8}\right\} \\ C &= m_2+\tfrac{1}{2}M_2\left\{1+\frac{3a^3b^3}{c^6}\left(1+\frac{3a^2}{c^2}+\frac{6a^4}{c^4}+\frac{11a^6}{c^6}\right)\right\} \end{aligned}\right\} \ldots\ldots(11).$$

These formulæ give the values of the coefficients as far as c^{-12}.

* There seems to be an error in equation (17) of Mr. Hicks' paper. I find

$$\frac{\mu_1}{\mu_0} = \tfrac{1}{2}\left\{\left(\frac{ab}{c^2-b^2}\right)^3-\frac{a^3b^3}{2c^2\,(c^2-b^2)^2}\right\},$$

which is of the order c^{-6}. The next ratio μ_2/μ_0 is of the order c^{-12}, and the above expression for μ_1/μ_0 agrees with (10) as far as c^{-10}, as ought to be the case.

5. If a sphere is projected in a liquid which is bounded by a fixed plane, we must put $a = b$, $u_1 = -u_2 = u$, $v_1 = v_2 = v$; then

$$2T = (A+B)\,u^2 + (A'+B')\,v^2,$$

and, if higher powers than c^{-6} be neglected,

$$A+B = m + \tfrac{1}{2}M\left(1 + \frac{3a^3}{c^3} + \frac{3a^6}{c^6}\right),$$

$$A'+B' = m + \tfrac{1}{2}M\left(1 + \frac{3a^3}{2c^3} + \frac{3a^6}{4c^6}\right),$$

where $c/2$ is the distance of the sphere from the plane. Lagrange's equation

$$\frac{d}{dt}\frac{dT}{du} - 2\frac{dT}{dc} = 0$$

gives $$(A+B)\,\dot{u} = v^2\frac{d}{dc}(A'+B') - u^2\frac{d}{dc}(A+B).$$

Also, since the momentum parallel to the plane is constant,

$$(A'+B')\,v = \text{const.} = G.$$

Let V be the resultant velocity of the sphere, θ the angle which its direction makes with the normal to the plane, then

$$(A+B)\,\dot{u} = V^2\left\{\sin^2\theta\,\frac{d}{dc}(A'+B') - \cos^2\theta\,\frac{d}{dc}(A+B)\right\}$$

$$= \frac{9MV^2a^3}{2c^4}\left\{\left(1+\frac{2a^3}{c^3}\right)\cos^2\theta - \tfrac{1}{2}\left(1+\frac{a^3}{c^3}\right)\sin^2\theta\right\}.$$

If, therefore, $$\tan\alpha = \sqrt{\frac{2\,(c^3+2a^3)}{c^3+a^3}},$$

it follows that, whenever the direction of motion makes with the normal to the plane an angle which is $< \alpha$ or $> \pi-\alpha$, the sphere will be repelled from the plane; but, whenever this angle lies between α and $\pi-\alpha$, the sphere will be attracted. Also, since $A'+B'$ increases as c diminishes, the velocity parallel to the plane will be accelerated when the direction of motion lies between α and $\pi-\alpha$; and retarded when this direction makes with the normal an angle $< \alpha$ or $> \pi-\alpha$. If, therefore, the sphere be projected parallel to the plane, it will ultimately strike it.

Mr. Hicks has elsewhere shown,* that in the case of a cylinder

* *Quarterly Journal*, Vol. XVI., p. 195.

$\alpha = \pi/4$, and we have shown that in the case of a sphere $\alpha > \pi/4$. The discussion of the subsequent motion of a sphere projected in any given direction in a liquid bounded by a fixed plane, can be carried on in the same manner as in the corresponding case of a cylinder, but it must be recollected that the preceding values of the coefficients will not give correct results if the sphere gets too close to the plane.

6. In order to obtain, by the preceding method, the potential when two charged spheres are placed in a field of electric force, we must first expand the potential of the field into two convergent series of spherical harmonics, whose origins are the centres of the two spheres respectively, and then obtain the potential corresponding to each term separately and add the results.

Let us illustrate this by considering a field whose potential is $L+Mz$.

Let E_1, E_2 be the charges, and V the potential of the two spheres; A and B the constant values of the potential of the spheres and field at the surface of the two spheres; then, near A,

$$V = A - L\frac{a}{r} - \frac{Ma^3P_1}{r^2} + A_0\left(1-\frac{a}{r}\right) + A_1\left(\frac{r}{a}-\frac{a^2}{r^2}\right)P_1$$

$$+ \ldots A_n\left(\frac{r^n}{a^n}-\frac{a^{n+1}}{r^{n+1}}\right)P_n,$$

also
$$\frac{E_1}{a} + L = -A_0,$$

whence

$$V = A - L - \frac{Ma^3P_1}{r^2} - \frac{E_1}{a}\left(1-\frac{a}{r}\right) + \Sigma_1^{\infty} A_n\left(\frac{r^n}{a^n}-\frac{a^{n+1}}{r^{n+1}}\right)P_n,$$

Similarly, near B,

$$V = B - L + Mc - \frac{Mb^3P_1'}{r'^2} - \frac{E_2}{b}\left(1-\frac{b}{r'}\right) + \Sigma_1^{\infty} B_n\left(\frac{r'^n}{b^n}-\frac{b^{n+1}}{r'^{n+1}}\right)P_n'.$$

And the successive coefficients must be determined by putting $m = 0$ in (1) and (2), in the manner already explained.

I have elsewhere pointed out that, if ψ be the current function when a solid of revolution is moving parallel to its axis with velocity V, then $2\psi\rho^{-1}\sin\chi$ is the potential of the induced charge when the

solid is placed in a field of electric force whose potential is

$$-V\rho \sin \chi,*$$

where z, ρ, χ are cylindrical coordinates.

The value of ψ for a doublet of strength m, whose axis coincides with that of z is $m\rho^2/r^3$, and therefore, when the spheres are moving along the line of centres, the complete value of ψ can be obtained by the method of images. Hence the potential can be obtained when the spheres are placed in a uniform field of force, which is perpendicular to a fixed plane which passes through their centres.

Second Note on Elliptic Transformation Annihilators.

By John Griffiths, M.A.

[*Read May 12th*, 1887.]

If P and Q be algebraic functions of x, involving the constant k, such that $y = \dfrac{P}{Q}$ leads to the transformation

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}} = \frac{M dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}},$$

where

$$M^2 = \frac{nkk'^2}{\lambda\lambda'^2} \frac{d\lambda}{dk},$$

then there are two operators Ω and O which reduce $P \div Q$ and $\lambda P \div Q$, respectively, to zero, *i.e.*, which give

$$\Omega \frac{P}{Q} = 0, \quad O \frac{\lambda P}{Q} = 0.$$

If the partial differentiation operators $\dfrac{d}{dk}$ and $\dfrac{d}{dx}$ be denoted by ∂_k and ∂_x, the general forms of Ω, O are

$$\Omega = nkk'^2\partial_k + \sqrt{1-x^2 \,.\, 1-k^2x^2}\; G(x, k)\, \partial_x,$$

$$O = nkk'^2\partial_k + \sqrt{1-x^2 \,.\, 1-k^2x^2}\; g(x, k)\, \partial^x,$$

where n is a number.

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* *Proc. Camb. Phil. Soc.*, Vol. VI., p. 7.

As G and g are, so far, not determined, it is clear that the above assumptions with regard to the forms of Ω and O are legitimate. They imply no more than the fact that $\frac{dy}{dx}$, $\frac{dy}{dk}$, $\frac{d(\lambda y)}{dk}$ must be functions of x and k, and that we write the ratios

$$\frac{dy}{dk} \div \frac{dy}{dx}, \quad \frac{d(\lambda y)}{dk} \div \frac{dy}{dx},$$

in the forms G and g in question. It is thus seen that $G(x, k)$ and $g(x, k)$ are of great importance in the theory of transformation, and I propose accordingly to investigate some of their properties.

Sect. 1.—If $\Omega y = 0$, $O\lambda y = 0$, and

$$G(x, k) - g(x, k) = \frac{M\lambda'^2 PQ}{\sqrt{Q^2 - P^2 \,.\, Q^2 - \lambda^2 P^2}},$$

where $\lambda^2 + \lambda'^2 = 1 = k^2 + k'^2$, and $M^2 = \frac{nkk'^2}{\lambda\lambda'^2} \frac{d\lambda}{dk}$,

then there must be a transformation

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}} = \frac{M\,dx}{\sqrt{1-x^2 \,.\, 1-k^2 x^2}}.$$

This result is easily proved. We have, by subtraction,

$$\Omega - O = \sqrt{1-x^2 \,.\, 1-k^2 x^2}\,\{G(x, k) - g(x, k)\}\,\partial_x,$$

and, therefore,

$$(\Omega - O)\,\lambda y = \sqrt{1-x^2 \,.\, 1-k^2 x^2}\,\{G(x, k) - g(x, k)\}\,\lambda \frac{dy}{dx},$$

i.e.,

$$y\Omega\lambda = \sqrt{1-x^2 \,.\, 1-k^2 x^2}\,\frac{M\lambda'^2 y}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}}\,\lambda \frac{dy}{dx},$$

or

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}} = \frac{M\,dx}{\sqrt{1-x^2 \,.\, 1-k^2 x^2}};$$

if $\Omega\lambda = M^2\lambda\lambda'^2$, which gives

$$M^2\lambda\lambda'^2 = nkk'^2 \frac{d\lambda}{dk}.$$

Sect. 2.—*Deduction of possible forms of G and g.*

The condition

$$G(x, k) - g(x, k) = \frac{M\lambda'^2 PQ}{\sqrt{Q^2 - P^2 \,.\, Q^2 - \lambda^2 P^2}}$$

is satisfied by taking

$$G(x,k) = X(x,k) + \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\left\{R \div (1-k^2x^2)^{\frac{1}{2}n}\right\},$$

$$g(x,k) = X(x,k) + \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\left\{S \div (1-k^2x^2)^{\frac{1}{2}n}\right\},$$

where $R = \sqrt{Q^2-\lambda^2P^2}$, $S = \sqrt{Q^2-P^2}$, and $X(x,k)$ is an arbitrary function of x and k. In this case we have, in fact,

$$G(x,k) - g(x,k) = \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\frac{R}{S}$$

$$= \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\sqrt{\frac{1-\lambda^2y^2}{1-y^2}},$$

or $$G(x,k) - g(x,k) = \sqrt{1-x^2 \cdot 1-k^2x^2}\left\{\frac{1}{1-y^2} - \frac{\lambda^2}{1-\lambda^2y^2}\right\} y\,\frac{dy}{dx}$$

$$= \frac{M\lambda'^2 y}{\sqrt{1-y^2 \cdot 1-\lambda^2y^2}}.$$

The reason for choosing these particular forms will become more apparent when we have considered the corresponding transformations for Θ-functions.

When n is a square number, the above will include the case of complete multiplication by $\sqrt{n}$. For example, let

$$n=4, \quad P = x\sqrt{1-x^2 \cdot 1-k^2x^2}, \quad Q = 1-k^2x^4, \quad \lambda = k, \quad M = 2,$$

$$R = \sqrt{Q^2-k^2P^2} = 1-2k^2x^2+k^2x^4,$$

$$S = \sqrt{Q^2-P^2} = 1-2x^2+k^2x^4,$$

$$G(x,k) = \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\left\{\frac{1-2k^2x^2+k^2x^4}{(1-k^2x^2)^2}\right\}$$

$$= \frac{4k^2k'^2x^3}{1-2k^2x^2+k^2x^4}\sqrt{\frac{1-x^2}{1-k^2x^2}},$$

$$g(x,k) = \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\left\{\frac{1-2x^2+k^2x^4}{(1-k^2x^2)^2}\right\}$$

$$= \frac{-4k'^2x}{1-2x^2+k^2x^4}\sqrt{\frac{1-x^2}{1-k^2x^2}},$$

$$\Omega = 4kk'^2\partial_k + 4\,\frac{k^2k'^2x^3(1-x^2)}{1-2k^2x^2+k^2x^4}\,\partial_x,$$

$$O = 4kk'^2\partial_k - 4\,\frac{k'^2x(1-x^2)}{1-2x^2+k^2x^4}\,\partial_x,$$

$$\frac{\Omega P}{P} = \Omega \log P = -\frac{4k^2k'^2x^2}{1-k^2x^2} + \frac{4k^2k'^2x^3(1-x^2)}{1-2k^2x^2+k^2x^4}\left\{\frac{1}{x} - \frac{x}{1-x^2} - \frac{k^2x}{1-k^2x^2}\right\},$$

or
$$\frac{\Omega P}{P} = \frac{4k^2k'^2x^2(1-2x^2)}{1-2k^2x^2+k^2x^4} - \frac{4k^2k'^2x^2}{1-k^2x^2}\left\{1 + \frac{k^2x^2(1-x^2)}{1-2k^2x^2+k^2x^4}\right\}$$

$$= \frac{4k^2k'^2x^2(1-2x^2)}{1-2k^2x^2+k^2x^4} - \frac{4k^2k'^2x^2}{1-2k^2x^2+k^2x^4}$$

$$= -\frac{8k^2k'^2x^4}{1-2k^2x^2+k^2x^4}.$$

$$\frac{\Omega Q}{Q} = \Omega \log Q = \Omega \log (1-k^2x^4) = -\frac{8k^2k'^2x^4}{1-k^2x^4} - \frac{16k^4k'^2x^8(1-x^2)}{(1-k^2x^4)(1-2k^2x^2+k^2x^4)},$$

i.e.,
$$\frac{\Omega Q}{Q} = -\frac{8k^2k'^2x^4}{1-2k^2x^2+k^2x^4},$$

therefore
$$\frac{\Omega P}{P} = \frac{\Omega Q}{Q}.$$

Similarly we may verify the equation

$$\frac{O\lambda P}{\lambda P} = \frac{OQ}{Q}.$$

Sect. 3.—*Transformation of* Θ*-functions.*

Taking the G functions to be

$$G(x,k) = X(x,k) + \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\left\{R \div (1-k^2x^2)^{\frac{1}{4}n}\right\},$$

$$g(x,k) = X(x,k) + \sqrt{1-x^2 \cdot 1-k^2x^2}\,\frac{d}{dx}\log\left\{S \div (1-k^2x^2)^{\frac{1}{4}n}\right\},$$

and
$$M^2 = \frac{nkk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk},$$

we have seen that in case $\Omega y = 0$ and $O\lambda y = 0$ there must be a transformation

$$\frac{dy}{\sqrt{1-y^2 \cdot 1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2 \cdot 1-k^2x^2}},$$

or, say,
$$v = M(u+u_0),$$

where
$$v = \int_0^y \frac{dy}{\sqrt{1-y^2 \cdot 1-\lambda^2y^2}}, \quad u = \int_0^x \frac{dx}{\sqrt{1-x^2 \cdot 1-k^2x^2}}.$$

The corresponding Θ-formulæ are

$$\Theta\left\{M(u+u_0),\lambda\right\} = C_1 e^{\mu u^2+\nu u}\,\Theta^n(u+K)\,e^{\int^u G(\mathrm{sn}\,u)\,du}\left(\frac{Q(\mathrm{sn}\,u)}{R(\mathrm{sn}\,u)}\right),$$

$$\Theta\left\{M(u+u_0)+\Lambda,\lambda\right\} = C_2 e^{\mu u^2+\nu u}\,\Theta^n(u+K)\,e^{\int^u G(\mathrm{sn}\,u)\,du},$$

$$\Theta\left\{M(u+u_0)+\Lambda+i\Lambda',\lambda\right\} = C_3 e^{\mu u^2+\nu_1 u}\,\Theta^n(u+K)\,e^{\int^u g(\mathrm{sn}\,u)\,du},$$

where C_1, C_2, C_3, μ, ν, ν_1 are k-functions.

It is not very difficult to establish these results by means of the equation

$$\frac{d}{dk}\,\mathrm{sn}\,(u,k) = -\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{kk'^2}\left\{Z(u+K)+\left(\frac{E}{K}-k'^2\right)u\right\},$$

and when the function $X(\mathrm{sn}\,u,\,k)$ may be neglected they reduce to the simple forms

$$\Theta\left\{M(u+u_0),\lambda\right\} = C_1 e^{\mu u^2+\nu u}\,\Theta^n u\,.\,Q(\mathrm{sn}\,u),$$

$$\Theta\left\{M(u+u_0)+\Lambda,\lambda\right\} = C_2 e^{\mu u^2+\nu u}\,\Theta^n u\,.\,R(\mathrm{sn}\,u),$$

$$\Theta\left\{M(u+u_0)+\Lambda+i\Lambda',\lambda\right\} = C_3 e^{\mu u^2+\nu_1 u}\,\Theta^n u\,.\,S(\mathrm{sn}\,u)\,;$$

where $R \equiv \sqrt{Q^2-\lambda^2P^2}$ and $S \equiv \sqrt{Q^2-P^2}$.

The final conclusion arrived at may be briefly stated as follows:

For a transformation

$$\Theta\left\{M(u+u_0),\lambda\right\} = Ce^{\mu u^2+\nu u}\,\Theta^n u\,.\,Q(\mathrm{sn}\,u),$$

where

$$M^2 = \frac{nkk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk},$$

the annihilators Ω and O, viz.,

$$\Omega = nkk'^2\partial_k + \sqrt{1-x^2\,.\,1-k^2x^2}\;G(x,k)\,\partial_x,$$

$$O = nkk'^2\partial_k + \sqrt{1-x^2\,.\,1-k^2x^2}\;g(x,k)\,\partial_x,$$

wherein G and g have the values

$$G(x,k) = \sqrt{1-x^2\,.\,1-k^2x^2}\,\frac{d}{dx}\log\left\{R\div(1-k^2x^2)^{\frac{1}{4}n}\right\},$$

$$g(x,k) = \sqrt{1-x^2\,.\,1-k^2x^2}\,\frac{d}{dx}\log\left\{S\div(1-k^2x^2)^{\frac{1}{4}n}\right\},$$

are not only necessary but *sufficient*.

For instance, we have seen that Ω and O are both satisfied in case

$$n=4,\quad P = x\sqrt{1-x^2\,.\,1-k^2x^2},\quad Q = 1-k^2x^4,$$

so that $$\Theta(2u) = C\Theta^4 u\,(1-k^2\,\mathrm{sn}^4\,u),$$

since $$\Theta(u+2mK) = \Theta(u) = \Theta(-u),$$

and $$\mathrm{sn}^2(u+2mK) = \mathrm{sn}^2\,u,$$

if m be any number.

In the more general case where $G(x, k)$ is of the form

$$G(x, k) = X(x, k) + \sqrt{1-x^2 \,.\, 1-k^2x^2}\,\frac{d}{dx}\log\left\{R \div (1-k^2x^2)^{\frac{1}{2}n}\right\},$$

we have $\Theta\left\{M(u+u_0)+\Lambda, \lambda\right\} = Ce^{\mu u^2+\nu u}\,\Theta^n(u+K)\,e^{\int^u G(\mathrm{sn}\,u)\,du}$,

or $$\Theta\left\{M(u+u_0)+\Lambda, \lambda\right\} = Ce^{\mu u^2+\nu u}\,\Theta^n u\,e^{\int^u X(\mathrm{sn}\,u)\,du}\,R;$$

but $\int_0^u X(\mathrm{sn}\,u)\,du$ is not necessarily integrable.

For instance, if

$$P = x\Delta a + a\Delta x, \quad Q = 1-k^2a^2x^2,$$

where $$\Delta x = \sqrt{1-x^2 \,.\, 1-k^2x^2},$$

we have a transformation $\nu = u+u_0$ (mod. k), but not necessarily

$$\Theta(u+u_0+K) = Ce^{\nu u}\Theta u \times \text{an algebraic function of } \mathrm{sn}\,u.$$

Sect. 4.—*Direct method of finding the* Θ*-formulæ.*

Taking the annihilators to be Ω and O, and writing their values in the forms

$$\Omega = nkk'^2\partial_k + \sqrt{1-x^2 \,.\, 1-k^2x^2}\,G(x, k)\,\partial_x,$$

$$O = nkk'^2\partial_k + \sqrt{1-x^2 \,.\, 1-k^2x^2}\,g(x, k)\,\partial_x,$$

we have seen (Sect. 1) that, if $\Omega y = 0$, $O\lambda y = 0$, and

$$G(x, k) - g(x, k) = \frac{M\lambda'^2 PQ}{\sqrt{Q^2-P^2 \,.\, Q^2-\lambda^2P^2}},$$

there must be a transformation

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}},$$

where $$M^2 = \frac{nkk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk}.$$

It follows from this that $\Omega y = 0$, under the conditions in question, gives

$$nkk'^2\frac{dy}{dk} = -\sqrt{1-x^2 \,.\, 1-k^2x^2}\,G(x, k)\,\frac{dy}{dx},$$

or $$nkk'^2\frac{dy}{dk} = -M\sqrt{1-y^2.1-\lambda^2y^2}\,G(x,k).$$

We have then the two equations

$$\frac{dy}{dx} = M\sqrt{\frac{1-y^2.1-\lambda^2y^2}{1-x^2.1-k^2x^2}},$$

$$nkk'^2\frac{dy}{dk} = -M\sqrt{1-y^2.1-\lambda^2y^2}\,G(x,k),$$

which lead us immediately to the Θ-formulæ.

If the first equation be differentiated with respect to k and the second with respect to x, the result

$$\sqrt{1-x^2.1-k^2x^2}\frac{dG}{dx} - \frac{M^2\lambda'^2}{1-\lambda^2y^2} + \frac{nk'^2}{1-k^2x^2} + M^2\lambda'^2 - nk'^2 + \frac{nkk'^2}{M}\frac{dM}{dk} = 0$$

is obtained after some easy reductions, since

$$\frac{d^2y}{dk\,dx} = \frac{d^2y}{dx\,dk},$$

Hence, supposing that

$$\frac{dy}{\sqrt{1-y^2.1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2.1-k^2x^2}}$$

gives $$v = M(u+u_0),$$

and introducing elliptic-function notation, we have

$$M^2\,\mathrm{dn}^2(v+\Lambda,\lambda) - n\,\mathrm{dn}^2(u+K) = \frac{dG(\mathrm{sn}\,u)}{du} + \text{a } k\text{-function},$$

viz., $$M^2\lambda'^2 - nk'^2 + \frac{nkk'^2}{M}\frac{dM}{dk}, \quad (\lambda \text{ being a function of } k),$$

or $$MZ\{M(u+u_0)+\Lambda,\lambda\} - nZ(u+K) = G(\mathrm{sn}\,u) + 2\mu u + \nu,$$

where μ and ν are κ or λ-functions, viz.,

$$\nu = MZ(Mu_0+\Lambda,\lambda) - G(0),$$

$$2\mu K + \nu = MZ(Mu_0+MK+\Lambda,\lambda) - G(1),$$

i.e., finally,

$$\Theta\{M(u+u_0)+\Lambda,\lambda\} = Ce^{\mu u^2+\nu u}\Theta^n(u+K)\,e^{\int^u G(\mathrm{sn}\,u)\,du}.$$

The other formula,

$$\Theta\{M(u+u_0)+\Lambda+i\Lambda',\lambda\} \div e^{\mu u^2+\nu u}\Theta^n(u+K)\,e^{\int^u G(\mathrm{sn}\,u)\,du} = \text{constant},$$

is derived immediately from this and the relation

$$G(\operatorname{sn} u) - g(\operatorname{sn} u) = M\lambda'^2 \operatorname{sn}(v, \lambda) \div \operatorname{cn}(v, \lambda) \operatorname{dn}(v, \lambda),$$

where $$v = M(u + u_0),$$

or $$\int g(\operatorname{sn} u)\, du - \int G(\operatorname{sn} u)\, du = \log \operatorname{sn}(v + \Lambda, \lambda).$$

Sect. 5.—*On some relations between the functions $G(x, k)$ and $g(x, k)$.*

In those transformation equations wherein the changes of y, x into $\frac{1}{\lambda y}$, $\frac{1}{kx}$ occur, we must *à priori* expect the change of Ω into O. In fact, if we write $\frac{1}{kx}$ for x, then ∂_k and ∂_x become $\partial_k - \frac{x}{k}\partial_x$ and $-kx^2\partial_x$, respectively; so that the new Ω is

$$nkk'^2\left(\partial_k - \frac{x}{k}\partial_x\right) + \frac{\sqrt{1-x^2}.\sqrt{1-k^2x^2}}{kx^2} G\left(\frac{1}{kx}, k\right) kx^2\partial_x,$$

or $$nkk'^2\partial_k + \left\{\sqrt{1-x^2}.\sqrt{1-k^2x^2}\, G\left(\frac{1}{kx}, k\right) - nk'^2x\right\}\partial_x.$$

This will be identical with O if

$$G\left(\frac{1}{kx}, k\right) - g(x, k) = \frac{nk'^2x}{\sqrt{1-x^2}.\sqrt{1-k^2x^2}}.$$

As illustrations of these results we may cite (1) Jacobi's n-transformations arising from $y = P \div Q$, wherein

$$P = x(1, x^2)^{\frac{1}{2}(n-1)} \text{ and } Q = (1, x^2)^{\frac{1}{2}(n-1)};$$

n being an odd number.

(2) The corresponding n-transformations when

$$P = x\sqrt{1-x^2}\,(1, x^2)^{\frac{1}{2}(n-2)}, \quad Q = \sqrt{1-k^2x^2}\,(1, x^2)^{\frac{1}{2}(n-2)};$$

n being an even number.

Similarly, in the transformation equations wherein x is changed into $\frac{1}{kx}$, while y remains unaltered, we must have an unaltered Ω.

This gives $$G\left(\frac{1}{kx}, k\right) - G(x, k) = \frac{nk'^2x}{\sqrt{1-x^2}.\sqrt{1-k^2x^2}}.$$

As an example, I notice the equation

$$y = \frac{x(1, x^2)^{\frac{1}{2}n-1}}{(1, x^2)^{\frac{1}{2}n}}; \quad n \text{ being even.}$$

For instance,

$$y = \frac{(1+k)\,x}{1+kx^2},$$

when $n = 2$.

These relations between the G-functions, derived by a purely algebraic process, are simplified by the introduction of elliptic-function notation, and may be verified by means of the Θ-formulæ of Sect. 3. For instance, in Jacobi's transformations, we have

$$\Theta(Mu+\Lambda, \lambda) = C_2 e^{\mu u^2} \Theta^n(u+K)\, e^{\int^u G(\mathrm{sn}\,u)\,du},$$

$$\Theta(Mu+\Lambda+i\Lambda', \lambda) = C_3 e^{\mu' u^2 + \nu_1 u} \Theta^n(u+K)\, e^{\int_0^u g(\mathrm{sn}\,u)\,du},$$

where the modular relation $MK' = bi\Lambda + c\Lambda'$ must be such that, when u is changed into $u+iK'$ in the first of these equations, we have the second. This gives

$$G\,\mathrm{sn}(u+iK') - g(\mathrm{sn}\,u) + n\frac{d}{du}\log\left(\frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}\right) = 0,$$

i.e.,

$$G\,\mathrm{sn}(u+iK') - g(\mathrm{sn}\,u) = \frac{nk'^2\,\mathrm{sn}\,u}{\mathrm{cn}\,u\,\mathrm{dn}\,u}.$$

Another relation in these transformations is

$$G\left(\sqrt{\frac{1-x^2}{1-k^2x^2}}\right) - G(x) + nk^2x\sqrt{\frac{1-x^2}{1-k^2x^2}} = M\lambda^2\,\frac{P\sqrt{Q^2-P^2}}{Q\sqrt{Q^2-\lambda^2P^2}}.$$

In all the transformations of this note we must have

$$G(x) - g(x) = \frac{M\lambda'^2 PQ}{\sqrt{Q^2-P^2}\,.\,Q^2-\lambda^2P^2}. \qquad \text{(See Sect. 1.)}$$

Sect. 6.—*On the determination of the coefficients in a transformation equation by means of the annihilators* Ω *and* O.

If any of the functions P, Q, R, S contain a factor $1-\frac{x^2}{a^2}$, a term $\frac{\Omega\frac{x}{a}}{1-\frac{x^2}{a^2}}$ or $\frac{O\frac{x}{a}}{1-\frac{x^2}{a^2}}$ will appear in the equations

$$\frac{\Omega P}{P} = \frac{\Omega Q}{Q} = \frac{\Omega S}{S}, \quad \frac{O\lambda P}{\lambda P} = \frac{OQ}{Q} = \frac{OR}{R}.$$

It is thus seen that $\Omega\frac{x}{a}$ or $O\frac{x}{a}$ must have the factor $1-\frac{x^2}{a^2}$ in

question. Hence

$$\left(\Omega\frac{x}{a}\right)_{x=a}=0 \quad \text{or} \quad \left(O\frac{x}{a}\right)_{x=a}=0;$$

i.e., since $\Omega x=\sqrt{1-x^2.1-k^2x^2}\,G(x,k)$, $\Omega a=nkk'^2\frac{da}{dk}=Oa$,

$$Ox=\sqrt{1-x^2.1-k^2x^2}\,g(x,k),$$

it follows that

$$nkk'^2\frac{da}{dk}\div\sqrt{1-a^2.1-k^2a^2}=G(a,k),$$

if $1-\frac{x^2}{a^2}$ be a factor of P, Q, or S, and

$$nkk'^2\frac{da}{dk}\div\sqrt{1-a^2.1-k^2a^2}=g(a,k)$$

when the said factor appears in either P, Q, or R.

If we take $y=P\div Q$, where

$$P=Mx\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x^2}{\beta^2}\right)\ldots,$$

$$Q=(1-k^2a^2x^2)(1-k^2\beta^2x^2)\ldots,$$

and apply the annihilator Ω, then

$$\begin{aligned}nkk'^2\left(\frac{dy}{dk}\right)_{x=a}&=-\sqrt{1-a^2.1-k^2a^2}\,G(a,k)\left(\frac{dy}{dx}\right)_{x=a}\\&=-\sqrt{1-a^2.1-k^2a^2}\,G(a,k)\,M\div\sqrt{1-a^2.1-k^2a^2}\\&=-MG(a,k).\end{aligned}$$

Now $$\left(\frac{dy}{dk}\right)_{x=a}=2M\frac{da}{dk}\frac{\left(1-\frac{a^2}{\beta^2}\right)\left(1-\frac{a^2}{\gamma^2}\right)\ldots}{Q_a},$$

where Q_a is the result of writing $x=a$ in Q. Hence

$$\begin{aligned}\frac{2M\frac{da}{dk}\left(1-\frac{a^2}{\beta^2}\right)\left(1-\frac{a}{\gamma^2}\right)\ldots}{Q_a}&=-MG(a,k)\div nkk'^2\\&=-M\frac{da}{dk}\div\sqrt{1-a^2.1-k^2a^2},\end{aligned}$$

or $$\frac{1}{\sqrt{1-a^2.1-k^2a^2}}+\frac{2\left(1-\frac{a^2}{\beta^2}\right)\left(1-\frac{a^2}{\gamma^2}\right)\ldots}{(1-k^2a^4)(1-k^2a^2\beta^2)(1-k^2a^2\gamma^2)}=0.$$

This relation can be verified at once by means of the equation

$$\left(\frac{dy}{dx}\right)_{x=a} = M \div \sqrt{1-a^2 . 1-k^2a^2}.$$

Again, since

$$G(x, k) = \sqrt{1-x^2 . 1-k^2x^2}\,\frac{d}{dx}\log\frac{R}{(1-k^2x^2)^{\frac{1}{2}n}}$$

$$= \sqrt{1-x^2 . 1-k^2x^2}\,\frac{d}{dx}\log\frac{Q\sqrt{1-\lambda^2y^2}}{(1-k^2x^2)^{\frac{1}{2}n}},$$

we have $G(a, k) = \sqrt{1-a^2 . 1-k^2a^2}\left\{\frac{nk^2a}{1-k^2a^2} + \left(\frac{\frac{dQ}{dx}}{Q}\right)_{x=a}\right\}$,

i.e., $nkk'^2\frac{da}{dk} = 1-a^2 . 1-k^2a^2\left\{\frac{nk^2a}{1-k^2a^2} - \frac{2k^2a^3}{1-k^2a^4} - 2k^2a\,\Sigma\frac{\beta^2}{1-k^2a^2\beta^2}\right\}$.

There are thus two systems of relations between the coefficients, one an integral, and the other a differential system. Another point of interest with regard to the annihilators is the relation between the G-functions corresponding to transformation and complete multiplication. For instance, let

$$y = \frac{P(x, k)}{Q(x, k)}, \quad Y = \frac{P_1(y, \lambda)}{Q_1(y, \lambda)}$$

be two transformation equations, so connected that one gives

$$\frac{dy}{\sqrt{1-y^2 . 1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2 . 1-k^2x^2}},$$

and the other $\frac{dY}{\sqrt{1-Y^2 . 1-k^2Y^2}} = \frac{N\,dy}{\sqrt{1-y^2 . 1-\lambda^2y^2}}$,

where $M^2 = \frac{nkk'^2}{\lambda\lambda'^2}\frac{d\lambda}{dk}, \quad N^2 = \frac{n\lambda\lambda'^2}{kk'^2}\frac{dk}{d\lambda}$;

then $G_m(x, k) = MG_1(y, \lambda) + nG(x, k)$,

and $g_m(x, k) = Mg_1(y, \lambda) + nG(x, k)$,

if $nkk'^2\frac{dy}{dk} + \sqrt{1-x^2 . 1-k^2x^2}\,G(x, k)\frac{dy}{dx} = 0,$

$$n\lambda\lambda'^2\frac{dY}{d\lambda} + \sqrt{1-y^2 . 1-\lambda^2y^2}\,G_1(y, \lambda)\frac{dY}{dy} = 0,$$

$$n^2kk'^2\frac{dY}{dk} + \sqrt{1-x^2 . 1-k^2x^2}\,G_m(x, k)\frac{dY}{dx} = 0,$$

$$n\lambda\lambda'^2\frac{d(kY)}{d\lambda}+\sqrt{1-y^2.1-\lambda^2y^2}\,g_1(y,\lambda)\,k\frac{dY}{dy}=0,$$

$$n^2kk'^2\frac{d(kY)}{dk}+\sqrt{1-x^2.1-k^2x^2}\,g_m(x,k)\,k\frac{dY}{dx}=0.$$

Again, in Jacobi's equations, the $G_m(x,k)$-function of complete multiplication by n is the sum of the $G(x,k)$-functions corresponding to the $n+1$ different transformations which arise from this multiplication. But I think the importance of the present theory has been sufficiently shown.

[When the previous note on the annihilators was written, the advantage of introducing the G-functions had not occurred to the writer. With this exception the notation of the two notes is identical. For first note, see *Proc. Lond. Math. Soc.*, Vol. XVIII., No. 290.]

June 9th, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

The President announced that the Council had that evening awarded the De Morgan Medal to Prof. Sylvester, F.R.S.

The following communications were made:—

Note on the Linear Covariants of the Binary Quintic: A. Buchheim.

The Motion of a Sphere in a Viscous Liquid: A. B. Basset.

On the Reversion of Series in connection with Reciprocants: Captain MacMahon, R.A.

Explanation of Illustrations of a Preliminary Note on Diameters of Cubics: J. J. Walker, F.R.S.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLII., No. 253.

"Proceedings of the Physical Society," Vol. VIII., Part IV.

"Educational Times," for June.

"Problems in Mechanism regarding Trains of Pulleys and Drums of Least Weight for a given Velocity Ratio," by Prof. H. Hennessy. (From *Proc. Roy. Soc.*, Vol. XLII., 8vo.)

"Problems on the Motion of Atoms," by J. K. Smythies, 4to; London, 1885.

"Bulletin de la Société Mathématique de France," T. xv., Nos. 3 and 4.

"Bulletin des Sciences Mathématiques," T. xi., Juin, 1887.

"Journal de l'École Polytechnique," 56 cahier.

"Annales de l'École Polytechnique de Delft," T. iii., 1re liv.; Leide, 1887.

"Atti della R. Accademia dei Lincei," Vol. iii., F. 7, 8; Roma, 1887.

"Nieuw Archief voor Wiskunde," Deel xiii., St. 2.

"Beiblätter zu den Annalen der Physik und Chemie," B. xi., St. 5.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1887, i.—xviii.

"Acta Mathematica," x., 1.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 33 and 34.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. vii., No. 5.

"Sur certaines Sommations Arithmétiques," par M. d'Ocagne, 4to pamphlet.

"Sur certaines Fonctions Symétriques: Application au calcul de la Somme des Puissances Semblables des Racines d'une Équation," par M. d'Ocagne, 4to pamphlet. (From the *Jornal de Sciencias.*)

"Sur les Péninvariants des Formes Binaires," par M. d'Ocagne, two copies, 4to pamphlets.

"Journal of the College of Science, Imperial University, Japan," Vol. i., Pt. ii.; Tokyo, Japan, 1887.

The following presents were received in the Recess:—

"Proceedings of the Royal Society," Vol. xlii., Nos. 254, 255, 256, 257.

"Educational Times," for July, August, September, and October.

"Cunningham Memoirs," No. 4, "Dynamics and Modern Geometry: a new Chapter in the Theory of Screws," by Sir R. S. Ball, 4to; Dublin, 1887.

"Greek Geometry, from Thales to Euclid," vii., by Prof. G. J. Allman, 8vo pamphlet (off-print from *Hermathena*, Vol. vi., pp. 269—278).

"The Great Trigonometrical Survey of India," Vol. iv*a*., 4to; Dehra Dun, 1886.

"Scientific Proceedings of the Royal Dublin Society," Vol. v., Parts 3, 4, 5, and 6, 8vo; Dublin, 1886, 1887.

"Scientific Transactions of the Royal Dublin Society," Vol. iii. (Series ii.); xi.—"On New Zealand Coleoptera, with Descriptions of New Genera and Species," by David Sharp, M.B.; xii.—"The Fossil Fishes of the Chalk of Mount Lebanon, in Syria," by J. W. Davis; xiii.—"On the Cause of Iridescence in Clouds," by G. Johnstone Stoney, 4to; Dublin, 1886, 1887.

"Johns Hopkins University Circulars," Vol. vi., No. 58.

"Annals of Mathematics," Vol. iii., No. 3, 4to; Charlottesville, Va.

"Smithsonian Report," 1885, Part i., 8vo; Washington, 1886.

"Leçons sur la Théorie Générale des Surfaces et les Applications Géométriques du Calcul Infinitésimal," par Gaston Darboux, 8vo; Paris, 1887.

"Bulletin des Sciences Mathématiques," Tome x.; "Tables des Matiéres et Noms d'Auteurs," Tome xi., July, August, September, October, 1887.

"Bulletin de la Société Mathématique de France," Tome xv., No. 5.

"Journal für die Reine und Angewandte Mathematik," Band 101, Heft iii., iv.; Band 102, Heft i.

"Beiblätter zu den Annalen der Physik und Chemie," Band xi., Stücke 6, 7, 8, 9.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. iii., Fasc. 9, 10, 11, 12, 13 (1 semestre); Vol. iv., Fasc. 1, 2, 3 (2 semestre).

"Acta Mathematica," x., 2, 3; Stockholm, 1887.

"Neue Beiträge zur Kenntniss des Baues und der Lebensgeschichte der Nematoden," von Rud. Leuckart, large 8vo; Leipzig, 1887.

"Ueber die Methode des Arithmetischen Mittels," erste Abhandlung, von C. Neumann, large 8vo; Leipzig, 1887.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahr. xxxi., Heft 3 and 4.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxi., 5me Livr.; Tome xxii., 1re Livr.

"Journal of the College of Science," Imperial University, Japan, Vol. i., Part iii., 4to; Tokyo, 1887.

"Proceedings of the Physical Society of London," Vol. ix., Part i., October, 1887.

"Memoirs of the National Academy of Sciences," Vol. iii., Part ii.; Washington, 1886.

Publications of the Morrison Observatory, Glasgow, Missouri, No. 1, 4to; Lynn, Mass., 1887.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," xix. to xxxix.

"Annali di Matematica," Tomo xv., Fasc. 2; Milano, Luglio, 1887.

"Rendiconti del Circolo Matematico di Palermo," Tomo i.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. vii., No. 6.

"Memorias de la Sociedad Cientifica 'Antonio Alzate,'" Tomo i., Nos. 1, 2; Mexico, 1887.

"Jahrbuch über die Fortschritte der Mathematik," Band xvi., Heft 3.

"Bollettino delle Pubblicazioni Italiane," Nos. 35 to 42; Firenze, 1887.

"Grundriss der Differential- und Integral-Rechnung," i. Theil—Differential-Rechnung, 8vo; Hanover, 1888 (*sic*), von M. Stegemann, Ph.D. (Dr. Ludwig Kiepert, editor).

"American Journal of Mathematics," Vol. ix., No. 4; Vol. x., No. 1.

"Association Française pour l'avancement des Sciences," Congrès de Nancy, 1886, "Sur le 25e Problème de 5e livre de Diophante et la Solution par Fermat," S. Roberts, F.R.S.

"The Conic Sections, with Solutions of Questions in London University and other Examination Papers," by George Heppel, M.A. (Baillière, Tindall, & Cox, 1887.) From the author.

Cabinet likenesses of Mr. W. J. Ibbetson, and of Prof. Sylvester, F.R.S. (two, one in academical dress), for the Society's album.

APPENDIX.

An account of the "Sine-Triple-Angle" circle (p. 3), will be found in the *Messenger of Mathematics*, Vol. XVI., pp. 125, 126; in *Mathesis*, Janvier, 1887, pp. 12—14, and in the *Educational Times*, for December, 1886 (in which number also its equation is proposed as Quest. 8838). In a subsequent number (April, 1887, Quest. 9046) further properties are indicated, to which we now add the following:—If A be joined to the in-centre cutting the sides EF, $E'F'$, in K, K' respectively, then $KE \cdot K'E' = KF \cdot K'F'$ and $AK \cdot AE' = AK' \cdot AF$, with like results for the other angles and sides. A simple geometrical construction is still a desideratum.

The "Circular Notes" (p. 141), occupy pp. 137—139 of the Appendix to Vol. XLVI. of the *Reprint from the Educational Times*. The "conjugate-Tucker" circles (p. 170), form part with the "orthocentroidal" circle (p. 179), of the Appendix to Vol. XVII. of the *Proceedings*, pp. 420—430. The "cosine-orthocentres" of a plane triangle and a cubic through them (p. 289), is printed in the *Messenger of Mathematics*, Vol. XVII., pp. 97—103. These points, Prof. Neuberg writes, are, in Prof. de Longchamps' terminology, "Brocardians of the orthocentre" (*cf.* also *Mathesis*, Mai, 1887, Note, p. 105).

Dr. Wolstenholme's "Note on a Tetrahedron" (p. 289) is given otherwise as a question (7459) in the *Educational Times*, of which a solution is given by the Proposer on pp. 86, 87, of Vol. XLVII. of the *Reprint*.

The following notelet on "Conics connected by the Invariant Relation $\Theta\Theta' = \Delta\Delta'$," was offered at the same meeting in the form:—"If u, v be two such conics, v may be written in either of the forms

$$(1)\quad 4u = \left(x_1\frac{\partial u}{\partial x} + y_1\frac{\partial u}{\partial y} + z_1\frac{\partial u}{\partial z}\right)^2 \Big/ u_1 + \left(x_2\frac{\partial u}{\partial x} + y_2\frac{\partial u}{\partial y} + z_2\frac{\partial u}{\partial z}\right)^2 \Big/ u_2,$$

$$(2)\quad 4u\left(x_2\frac{\partial u_1}{\partial x_1} + y_2\frac{\partial u_1}{\partial y_1} + z_2\frac{\partial u_1}{\partial z_1}\right) = \left(x_1\frac{\partial u}{\partial x} + y_1\frac{\partial u}{\partial y} + z_1\frac{\partial u}{\partial z}\right)\left(x_2\frac{\partial u}{\partial x} + y_2\frac{\partial u}{\partial y} + z_2\frac{\partial u}{\partial z}\right),$$

where $(x_1y_1z_1)$, $(x_2y_2z_2)$ are any two points, u_1, u_2 the values of u at these points."

Mr. Basset's communication (p. 388), is embodied in a Memoir read before the Royal Society (November 24th, 1887), and Mr. Walker's "explanation" (p. 388), had reference to a paper "received April 21st, 1887," and of which a short abstract is given in Vol. XLII., pp. 334, 335, of the *Proceedings of the Royal Society*.

The substance (with necessary modifications) of Sir James Cockle's paper "On Binomial Biordinals," read April 6th, 1882, and afterwards withdrawn (see *Proceedings*, Vol. XIII.), appears in the *Philosophical Magazine*, for September, 1887 (Vol. XXIV., pp. 249—252).

Mr. Frank Scott Haydon, eldest son of the well-known painter Benjamin Robert Haydon, was born December 12th, 1822 (see father's Life), and received his name in honour of his godfather, Sir Walter Scott. Originally intended for the Church, he was educated at Jesus College, Cambridge, "where he became a good classic and an excellent mathematician."* Having abandoned his original intention and being left without resources by his father's suicide, he was, in 1846, nominated by Sir R. Peel to a clerkship in the Customs. The duties of this post proving uncongenial, he was soon transferred to the Public Record Office, where he remained until his death, and gradually rose to the rank of Assistant-Keeper. He edited, for the Master of the Rolls, the *Eulogium Historiarum*, a work "embodying some important contemporary evidence respecting the campaigns of the Black Prince." "For some years past, symptoms of his mental disorder were painfully apparent, in the form of violently excitable moods, a morbid taste for repulsive themes, and strange lapses of memory . . . . it was always possible to quiet his most turbulent outbursts and restore for a time his mental balance by proposing a mathematical problem for his solution." He died, by his own hand, Saturday, October 29th, aged 64. He was elected a member of the Society, October 16th, 1865.

Further particulars may be gathered from the *Times*, of November 1st; *Athenæum*, of November 5th and 12th.

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* According to the statement of "H. G. H." in the *Academy*, November 19th, 1887. His name does not appear in either Tripos List, and a writer in the *Athenæum*, says, "he knocked himself up with hard work." In Romilly's *Graduati Cantabrigienses*, we find him entered "A.B., 1845."

The following results were arrived at early in June last, or when the notes were written out in October:—

ABC is a triangle of which O is the circumcentre, K the Lemoine-point, and Ω, Ω' the Brocard points. The joins of A, B, C with Ω, Ω' respectively meet the circumcircle in β, γ, α; γ', α', β'; thus forming two *uni-Brocardal triangles* (*i.e.*, two triangles which have each *one* Brocard point in common with the primitive triangle), in perspective with ABC.

$$\left.\begin{matrix}\beta\gamma,\ \gamma\alpha,\ \alpha\beta\\ \alpha\beta,\ \beta\gamma,\ \gamma\alpha\end{matrix}\right\}\ \text{cut } BC, CA, AB \left\{\begin{matrix}p,\ q,\ r;\\ p_1,\ q_1,\ r_1;\end{matrix}\right.$$

and

$$\left.\begin{matrix}\beta'\gamma',\ \gamma'\alpha',\ \alpha'\beta'\\ \gamma'\alpha',\ \alpha'\beta',\ \beta'\gamma'\end{matrix}\right\}\quad\text{,,}\quad\text{,,}\quad\left\{\begin{matrix}p',\ q',\ r';\\ p_1',\ q_1',\ r_1'.\end{matrix}\right.$$

From the construction, the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are equal to ABC, and the arcs $B\beta$, $C\gamma$, $A\alpha$; $C\gamma'$, $B\beta'$, $A\alpha'$, are all equal; Ω is the positive Brocard-point of ABC and the negative point of $\alpha\beta\gamma$, Ω' is the negative Brocard-point of ABC and the positive point of $\alpha'\beta'\gamma'$. Further, $\alpha\beta'$ is parallel to AB, $\beta\gamma'$ parallel to BC, $\gamma\alpha'$ parallel to AC. .(i.)

Omitting the respective constant moduli, the trilinear coordinates of the following points are

$$\left.\begin{matrix}\alpha = c^2,\ ab,\ -ac^3/(a^2+c^2)\\ \beta = -ba^3/(a^2+b^2),\ a^2,\ bc\\ \gamma = ca,\ -cb^3/(b^2+c^2),\ b^2\end{matrix}\right\},\quad\left.\begin{matrix}\alpha' = b^2,\ -ab^3/(a^2+b^2),\ ac\\ \beta' = ab,\ c^2,\ -bc^3/(b^2+c^2)\\ \gamma' = -ca^3/(a^2+c^2),\ bc,\ a^2\end{matrix}\right\}\quad\text{...(ii.),}$$

$$\left.\begin{matrix}p = 0,\ a^2c,\ b\,(b^2+c^2)\\ q = c\,(c^2+a^2),\ 0,\ ab^2\\ r = bc^2,\ a\,(a^2+b^2),\ 0\end{matrix}\right\},\quad\left.\begin{matrix}p' = 0,\ c\,(b^2+c^2),\ a^2b\\ q' = cb^2,\ 0,\ a\,(c^2+a^2)\\ r' = b\,(a^2+b^2),\ ac^2,\ 0\end{matrix}\right\}\quad\text{.....(iii.)}$$

The $$\angle CA\gamma = \omega = \angle A\gamma\alpha = \angle AC\alpha' = \angle\gamma'\alpha'C,$$

hence $$Aq = \gamma q \quad\text{and}\quad Cq' = \alpha'q',$$

and the four lines are equal by (i.), and

$$2R\sin(B-\omega) = A\gamma = 2Aq\cos\omega,$$

therefore
$$Aq = R\sin(B-\omega)/\cos\omega = Rb^2\sin B/(\lambda\cos\omega)$$
$$= b^3/K = Cq' = \&c. \text{ .....(iv.)}$$

Referring to I. x.,[1] we see that $b^3/K = [EE']$, *i.e.*, the intercept

[1] The "triplicate-ratio" circle, *Quar. Jour. of Math.*, Vol. xix., No. 76, is quoted as I., and the letters of this paper are enclosed in [ ].

made by the "T.R." circle on AC. In like manner, we have

$$\left.\begin{aligned} Bp' = Cp = a^3/K = [DD'] = p'\gamma' = p\beta \\ Ar' = Br = c^3/K = [FF'] = ar = \beta' r' \end{aligned}\right\} \quad \text{.....(v.)}$$

From the previous work we see that

$$\angle Aqa = 2\omega = \angle Ar'a' ;^{(2)}$$

hence $a\gamma$, $a'\beta'$ are antiparallels, as also are $a\beta'$, $a'\gamma$; $a'\gamma$, AB; $a\beta'$, AC; therefore circles will go round $qr_1r'q_1'$; $a'\beta'q_1'q$; $a'\gamma r_1r'$; and like results, of course, hold for the lines similarly situated with respect to the sides BC, BA; CA, CB .....(vi.)

Now $\angle qOr' = 180° - (A + 2\omega) = \angle Ar_1q$, therefore the circle $qr_1r'q$ passes through O, and in like manner the circles $p'p_1rr_1'$, $p_1'p_1q_1q'$ pass through O .....(vii.)

Reverting to (iv.), we get $qq' = b\,(1 - 2b^2/K) = 2abc\cos B/K =$ intercept made on CA by the cosine-circle .....(viii.)

The perpendicular from O on $a\beta'$ passes through the intersection (s say) of $a\beta$, $a'\beta'$, and through that of aB, $A\beta'$, and this last point is the in-centre of $as\beta'$; like results hold for the other sides ..... (ix.)

Now, join pqr, $p'q'r'$; then

$$K^2 \cdot qr_1^2 = b^6 + c^2\,(a^2+b^2)^2 - 2b^3c\,(a^3+b^2)\cos A$$
$$= a^2\,(a^2b^2 + b^2c^2 + c^3a^2) = a^2\lambda^2,$$

therefore $$qr = a\lambda/K = a\sec\omega/2, \quad \text{(I. xii.)}$$

and $$\sec\omega/2 = \rho/R,$$

where ρ is the "T.R." radius; hence circum-circles pqr, $p'q'r'$ are equal to the "T.R." circle, and the triangles are equal to one another and similar to ABC .....(x.)

Because the angles γ and C are equal, therefore p, q, γ, C are concyclic, hence

$$\angle pqC = \angle \beta\gamma C = \angle \beta AC,$$

therefore pq is parallel to $A\Omega B$, similarly $p'q'$ is parallel to $B\Omega'a'$, and so on, *i.e.*,

$$\left.\begin{matrix} pq \\ p'q' \end{matrix}\right\} \text{produced make angles with } \left.\begin{matrix} BA \\ AB \end{matrix}\right\} = \omega \text{ .....(xi.)}$$

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(2) These and many other results may be easily got by noting that $a\beta\gamma$, $a'\beta'\gamma'$ are got by turning any line, as OA, counter-clockwise or clockwise respectively through an angle 2ω.

This result should be compared with the similar property of the triangles $[DEF, D'E'F']$ in I. xii.

For the moment, let ρ' be the circumradius of aqr, then

$$a \sec \omega/2 = qr = 2\rho' \sin A = 2\rho \sin A, \text{ by (x.)},$$

therefore $\rho' = \rho$, and as $\angle qOr = 180° - A$, by (xiv.), therefore A, a, r, O, q are concyclic, and the circle is equal to the "T.R." circle; hence, taking account of the other similarly obtained circles, we see that we have now got

$$6+2+1 = 9 \text{ "T. R." circles} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xii.)}$$

Assume $$\angle qOq' = 2\theta,$$

then $$\tan\theta = abc \cos B/(RK \cos B) \text{ by (viii.)}$$
$$= 4\Delta/K = \tan\omega,$$

therefore $$\theta = \omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xiii.)}$$

Similarly pp', rr' subtend angles 2ω at O (*cf.* Note 2).

From (xiii.), we have $\angle OqC = 90° - \omega$, and $\angle pqC = A - \omega$, by (xi.), therefore

$$\angle Oqp = 90° - A,$$

i.e., O is the orthocentre of the triangles pqr, $p'q'r'$ .....(xiv.)

Lines through p, q, r, parallel to AC, BA, CB, intersect in ω_1', the negative Brocard-point of pqr; of which the coordinates are (because perpendicular from ω_1' on $BC = c^3 \sin B/K$) proportional to c^2/a, a^2/b, b^2/c; and lines through p', q', r', parallel to BA, BC, CA, intersect in ω_2', the positive Brocard-point of $p'q'r'$, and its coordinates are proportional to

$$b^2/a, \ c^2/b, \ a^2/c \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xv.)}.$$

The mid-point of the join of ω_1', ω_2', is given by

$$(b^2+c^2)/a, \ \ (c^2+a^2)/b, \ \ (a^2+b^2)/c,$$

i.e., it is the Lemoine point of the medial triangle of ABC [3] ...(xvi.), and the equation to $\omega_1'\omega_2'$ is

$$a\,(a^4 - b^2c^2)\,\alpha + \ldots + \ldots = 0 \ \ldots\ldots\ldots\ldots\ldots\text{(xvii.)},$$

and the equation to B_1C_1 ($A_1B_1C_1$ Brocard's first triangle) is

$$a\,(a^4 - b^2c^2)\,\alpha + b\,(c^4 - a^2b^2)\,\beta + c\,(b^4 - c^2a^2)\,\gamma = 0,$$

hence $\omega_1'\omega_2'$ intersects B_1C_1 on the median through A.

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[3] *Cf.* "The Symmedian point Axis," &c., *Quarterly Journal of Mathematics*, Vol. xx., No. 78.

If O_1, O_1', are the circumcentres of pqr, $p'q'r'$,

$$\angle O_1pB = C-\omega+90°-B,$$

therefore α-coordinate of $O_1 = R\sec\omega\,.\cos(B+\omega-C)/2$,

and $$\angle O_1'p'C = B-\omega+90°-C,$$

therefore α-coordinate of $O_1' = R\sec\omega\,.\cos(C+\omega-B)/2$,

therefore α-coordinate of mid-point of $O_1O_1' = R\cos(B-C)/2$;

hence this mid-point is the nine-point centre of ABC, and the radical axis of pqr, $p'q'r'$ is OH, where H is the orthocentre .....(xviii.)

The circles round $pq\gamma C$ and $p'q'\gamma'C$ have for equations

$$cK(a\beta\gamma+...+...) = (a\alpha+...+...)\left\{b^3a+a(b^2+c^2)\beta\right\},$$

$$cK(a\beta\gamma+...+...) = (a\alpha+...+...)\left\{b(c^3+a^2)\alpha+a^3\beta\right\}...(xix.),$$

hence their radical axis is the join of O and C; and the circumcentre is the radical centre of the six like circles. The radical axis of the pair $q'r'a'A$, $rp\beta B$ is

$$aca-bc\beta+(a^2-b^2)\gamma = 0,$$

i.e., it cuts AB at its mid-point and passes through $(a, b, -c)$; and similarly for the other pairs .....(xx.).

From (ii.) we see that $a\gamma$, $a'\gamma'$ intersect in B' upon the line

$$a\alpha(c^4+a^2b^2) = \gamma c(a^4+b^2c^2);$$

whence we find that the centre of perspective of ABC, $A'B'C'$ is

$$a\alpha/(a^4+b^2c^2) = \beta b/(b^4+c^2a^2) = \gamma c/(c^4+a^2b^2) \quad .....(xxi.).$$

If $a'\gamma$, $\beta'a$ meet in A'', and so on, we find that A'' lies on

$$c^3\beta = b^3\gamma,$$

hence the centre of perspective (S) of the triangles ABC, $A''B''C''$ is the point (a^3, b^3, c^3) which lies on the circum-Brocardal axis,[4] and is the inverse of the centre of perspective of the triangles ABC, $A_1B_1C_1$ .....(xxii.).

The perpendicular from a (or β') on AB

$$= 2c^3\Delta/\lambda^2 = c^3\sin 2\omega/K,$$

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[4] The point η of "the 'Cosine' Orthocentres," &c., *Messenger of Mathematics*, Vol. XVII., p. 97. If we produce $\beta'\gamma$, $\gamma'a$, $a'\beta$ then (*Proceedings*, Vol XV., p. 126), this point is the common pole of the three triangles.

and from S on the same line $= c^3 \tan 2\omega/(2K)$;

hence from S on $A''B' = c^3 \cos\omega \sin 3\omega/(K \cos 2\omega)$,

and the linear dimensions of the triangles are

$$\frac{a}{a''} = \frac{SA}{SA''} = \frac{\sin\omega}{\sin 3\omega} \quad \ldots\ldots\ldots\

hence the radical axis is given by

$$(b^2-c^2)\cos A\alpha+\ldots+\ldots=0,$$

and is therefore the line OH; *cf.* (xviii.)

The axes of perspective of ABC, $\alpha\beta\gamma$; ABC, $\alpha'\beta'\gamma'$, are given by

$$bc\,(a^2+b^2)\,\alpha+ca\,(b^2+c^2)\,\beta+ab\,(c^2+a^2)\,\gamma=0,$$

$$bc\,(a^2+c^2)\,\alpha+ca\,(b^2+a^2)\,\beta+ab\,(c^2+b^2)\,\gamma=0,$$

and these intersect on the circum-Brocardal axis .....(xxix.)

Since $Ar=r\beta=\alpha'r'=r'B$, &c., therefore

rO bisects $A\beta$ at right angles in δ, say,

and $q'O$ „ $A\gamma'$ „ δ' „

and the projection of $A\delta$ on AH

$$=R\sin(C+\omega)\sin(B+\omega)=\text{the projection of } A\delta' \text{ on } AH,$$

therefore $\delta\delta'$ is perpendicular to AH; *cf.* (i.).....(xxx.)

We now briefly consider some properties of the triangles $p_1q_1r_1$, $p_1'q_1'r_1'$. As in (iii.), we have

$$\left.\begin{matrix} p_1,\ 0,\ a^2\,(a^2+c^2),\ bc^3 \\ q_1,\ a^3c,\ 0,\ b^2\,(a^2+b^2) \\ r_1,\ c^2\,(b^2+c^2),\ ab^3,\ 0 \end{matrix}\right\},\quad \left.\begin{matrix} p_1',\ 0,\ b^3c,\ a^2\,(a^2+b^2) \\ q_1',\ b^2\,(b^2+c^2),\ 0,\ ac^3 \\ r_1',\ a^3b,\ c^2\,(c^2+a^2),\ 0 \end{matrix}\right\} \ldots(\text{xxxi.}),$$

and $$Aq_1'\,.\,Br_1'\,.\,Cp_1'=Ar_1\,.\,Bp_1\,.\,Cq_1.$$

The equations to pr_1, $p'q_1'$ are respectively

$$ab^4\alpha-bc^2\,(b^2+c^2)\,\beta+a^2c^3\gamma=0,$$

$$ac^4\alpha+a^2b^3\beta-b^2c\,(b^2+c^2)\,\gamma=0,$$

the former of which passes through Ω, and the latter through Ω', *i.e.*, Ω is the centre of perspective of pqr, $p_1q_1r_1$; and Ω' of $p'q'r'$, $p_1'q_1'r_1'$.

.....(xxxii.)

The respective axes of perspective coincide with those obtained in (xxix.).

Making use of Note (2), we see that the triangles $p_1q_1r_1$, $p_1'q_1'r_1'$ are equal. If

$$L\equiv b^4+b^2c^2+c^4,\quad M\equiv c^4+a^2c^2+a^4,\quad N\equiv a^4+a^2b^2+b^4,$$

we get

$$p_1q_1^2=a^6\left[a^2b^2M^2+(a^2+c^2)^2N^2-2ab\,(a^2+c^2)\,MN\cos C\right]/M^2N^2,$$

and $p_1'q_1'^2 = a^6\left[a^2c^2N^2+(a^2+b^2)^2M^2-2ac\,(a^2+b^2)\,MN\cos B\right]/M^2N^2$, therefore either

$$= a^6\left[a^2\,(b^2M^2+c^2N^2)+LMN-2b^2c^2MN\right]/M^2N^2\,;$$

the other sides are, of course, easily obtained by symmetry...(xxxiii.)

The area of either triangle is found to be, from (xxxi.),

$$a^2b^2c^2\,K\lambda^2\Delta/LMN.$$

If the sides p_1q_1, q_1r_1, r_1p_1 make angles θ_1, θ_2, θ_3 with BC, CA, AB; and $p_1'q_1'$, $q_1'r_1'$, $r_1'p_1'$ angles ϕ_1, ϕ_2, ϕ_3 with CA, AB, BC, then

$$\sin\theta_1\,.\,\sin\theta_2\,.\,\sin\theta_3 = \sin\phi_1\,.\,\sin\phi_2\,.\,\sin\phi_3\,.$$

The other Brocard and Lemoine points whose coordinates have not been obtained can easily be found geometrically, if we bear in mind Note (2). Other uni-Brocardal triangles can be derived from our figure, and if ω is a sub-multiple of π, the cycle of triangles will recur.

If through p, q, r we draw parallels to AB, BC, CA, and through p', q', r', parallels to CA, AB, BC, these lines cut the sides BC in D, D'; CA in E, E'; AB in F, F', as in the figure to (I.), *i.e.*, these six points lie on the "T.R." circle of ABC. Hence, as the "T.R." circle of ABC determines a "T.R." hexagon, so the "T.R." circles of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ determine other "T.R." hexagons of which pqr, $p'q'r'$, are respectively three points. This group of six "T.R." triangles seems to offer points of interest to the geometer.

We can only glance here at the extension to uni-Brocardal Harmonic Polygons. It is readily seen that such polygons are equal in all respects to the primitive figure.

In the case of the Harmonic quadrilateral, I get that the two in-quadrilaterals (corresponding to pqr, $p'q'r'$) have circum-circles equal to the First Lemoine (or "T.R.") circle of $ABCD$ (see figure to "Some Properties of a Quadrilateral, &c.," Appendix to *Reprint*, Vol. XLIV.), and that there are circles corresponding to those in (xii.) *supra*; hence in this case there are $8+2+1 = 11$ such circles. The general result would be $2n+3$ "T.R." circles.

In the case of the quadrilateral and of the polygon, as some of the sides of the primitive will contain two (or more) points corresponding to p, p', it is plain that the quadrilaterals, &c., will not be convex figures. The property of (iv.) and (v.) holds for the quadrilateral, but I have not, at present, pursued the analogy any further.

I have noted but few works this year bearing on the Brocardian Geometry. The principal is a paper "Sur les Polygones et les

Polyèdres Harmoniques," presented by MM. G. Tarry and J. Neuberg to the Association Française, at its Nancy Meeting (August 13th, 1886). Dr. Casey refers to this in his "Sequel to Euclid." M. de Longchamps sends copies of his paper "Sur un Nouveau Cercle remarquable du Plan d'un Triangle," and "Généralités sur la Géométrie du Triangle, les Points Réciproques et les Potentiels d'ordre p." These are offprints from his *Journal de Mathématiques*. M. Vigarié has contributed a Paper "Sur les points Complémentaires," to *Mathesis*, January, 1887. Dr. Lieber sends a number of the *Zeitschrift*. Mr. R. F. Davis communicates a "Note on the Brocardal Ellipse," and a "Geometrical Construction for the Brocard Angle, &c." to Vol. XLVII. of the "Reprint from the *Educational Times*." In this journal will be found numerous questions on the subject, the most noteworthy new contributor being Prof. A. M. Nash, of the Calcutta University. The Rev. T. C. Simmons' "New Method for the Investigation of Harmonic Polygons" has been published by the Society. The same writer has contributed several chapters, "On the recent Geometry of the Triangle," to the Rev. J. J. Milne's "Companion to Weekly Problem Papers," which is shortly to appear. The subject is treated in a manner well adapted for junior students, and is illustrated by many interesting results. M. Brocard has been moved from Montpellier to Grenoble, and is now chef-de-bataillon. Whilst we congratulate him on his promotion, we regret to find that his higher rank will leave him less time for the pursuit of mathematical research.

R. T.

November, 1887.

The following corrections should be made in the present volume :—

p. 179, line 8, *for* triangle, *read* circle.
p. 183, last line, in U', *for* u', u, *read* ω', ω.
p. 187, doubly accent the last term of Art. 29.
p. 187, last line, accent the last term.
p. 188, line 2, accent the first negative term.
p. 191, line 4, *for* r, *read* $\dot{r}$, and *for* $\dot{\pi} = -1$, *read* $\dot{\pi}' = -1$.
p. 192, Art. 52, line 3, accent the last term.
p. 195, Art. 63, in the first terms of both sides of [5] and [6], *for* x, *read* X.
p. 197, line 3, *for* V *read* $\dot{V}$.
p. 197, line 10, *insert* $= 0$ before the semicolon.
p. 197, Art. 70, line 6, *for* Θ'_3 *read* Θ_3.
p. 199, Art. 74, last line, *insert* $\dot{u}u$ after the last bracket.
p. 200, Art. 77, line 10, *for* $4x$, *read* $4x^2$, and mark the equation as (6).
p. 326, line 2 up, *read* "Schlæfli."
p. 360, line 10, *for* Two Spaces *read* Two Spheres.

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